A hint for Lionel Levine's hat puzzle

N wizards wear infinite stacks of hats of two colors, red and blue. The hats are visible to all, with the exception that wizards cannot see their own stack of hats. In each stack the hat colors were determined by the evil sultan using a fair coin. At the sultan's signal, each wizard writes down a positive integer. If every wizard's integer gives the position of a blue hat in their stack, all the wizards are released. Otherwise — if even one wizard's integer points to a red hat in their unseen stack — all the wizards are forced to serve the sultan for the rest of their lives. Clearly, if each wizard writes down a random integer, the probability of release is only $1/2^N$. Here's a hint for a strategy the wizards can use to hugely improve their odds:

- Instead of each wizard being focused on maximizing their own blue-hat hitting probability, as a group they bet on the truth of a mathematical proposition whose details they work out in advance.

- The proposition is a Boolean function $f(p_1, \ldots, p_N)$, where p_i is the position (a positive integer) of the lowest blue hat in the stack worn by wizard i. Recall that wizard i knows all the numbers $p_1, \ldots, p_N$ with the exception of p_i.

- The function f is designed to have a relatively high probability of being true given the sultan's random hat-coloring protocol.

- The function f also has the property that if true, then wizard i is able to determine a unique position p_i that makes it true.