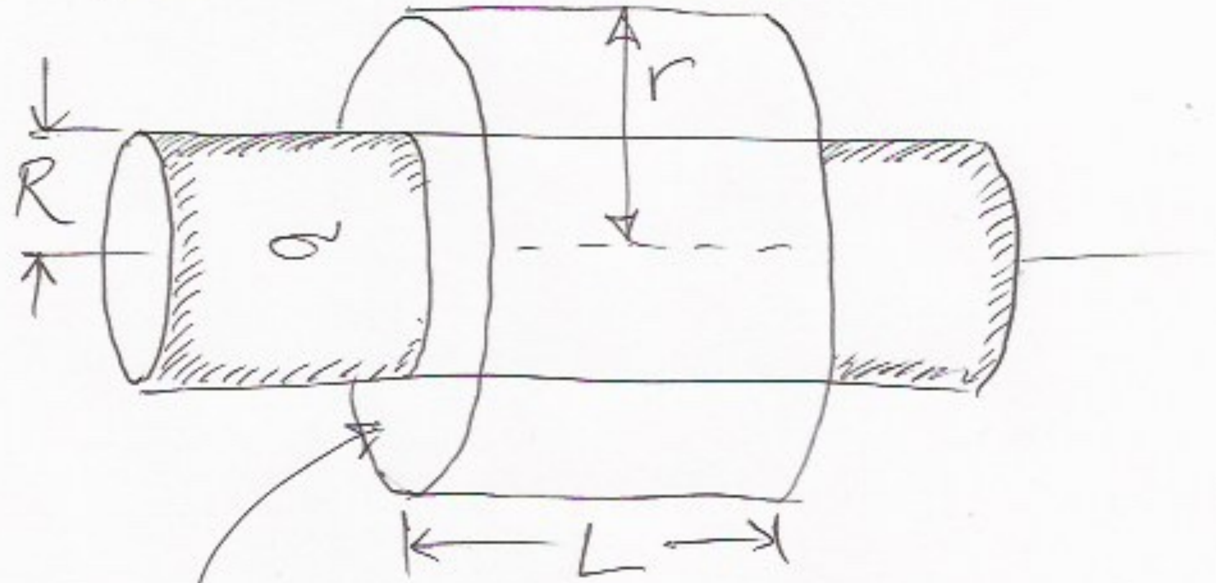
Assignment 9 Solutions



$S(r,L)$ = cylinder surface

by symmetry: $\vec{E} = E(r)\hat{r}$

Gauss: $\oint_{S(r,L)} \vec{E}\cdot d\vec{a} = E(r)\cdot 2\pi r L$

$$= \frac{Q_{enc.}}{\epsilon_0} = \begin{cases} 0, & r<R \\ \frac{2\pi r L \sigma}{\epsilon_0}, & r>R. \end{cases}$$

$$\Longrightarrow E(r) = \begin{cases} 0, & r<R \\ \frac{\sigma}{\epsilon_0}\frac{R}{r}, & r>R. \end{cases}$$

Since for $r>R$ the electric field has exactly the same form as that of an infinite line of charge, we can use the results from lecture in that region:

$$\lambda \to 2\pi R\sigma$$

$$\vec{F} = \begin{cases} 0, & r<R \\ -qvB\hat{r}, & r>R \end{cases}$$

$$B = \frac{\mu_0 I}{2\pi r}, \quad I = 2u\lambda = 4\pi u R\sigma$$

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1. The "double-curl" identity is an instance of the double-cross-product identity, where two of the vectors are the gradient operator: $\vec{\nabla} = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$

$$\vec{A} = \vec{\nabla}, \ \vec{B} = \vec{\nabla}, \ \vec{C} = \vec{V}$$

$$(*) \quad \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - (\vec{A} \cdot \vec{B})\vec{C}$$

Note that the right-hand-side has $\vec{C}$ always on the right, so the derivatives in $\vec{A}$ & $\vec{B}$ will act on it ($\vec{V}$).

To check (*), just work out the z-component on both sides, since the other two components are related by symmetry:

$$(A_x \hat{x} + A_y \hat{y} + \cdots) \times \Big((B_y C_z - B_z C_y)\hat{x} + (B_z C_x - B_x C_z)\hat{y} + \cdots \Big)$$

$$= \Big(A_x (\overset{①}{B_z C_x} - \overset{②}{B_x C_z}) - A_y (\overset{③}{B_y C_z} - \overset{④}{B_z C_y}) \Big)\hat{z}$$

$$+ \cdots$$

$$\vec{B}(\vec{A}\cdot\vec{C}) - (\vec{A}\cdot\vec{B})\vec{C}$$

$$= \Big(B_z \big(\overset{\textcircled{1}}{A_x C_x} + \overset{\textcircled{4}}{A_y C_y} + \cancel{A_z C_z} \big) \Big\updownarrow \text{cancel}$$

$$- \big(\underset{\textcircled{2}}{A_x B_x} + \underset{\textcircled{3}}{A_y B_y} + \cancel{A_z B_z} \big) C_z \Big) \hat{z}$$

$$+ \cdots$$

The four terms that do not cancel exactly match the four terms on the previous page.

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2. Let φ_0 be the point-charge solution:

$$\varphi_0(\vec{r};\vec{r}\,') = \frac{q}{4\pi\epsilon_0 |\vec{r}-\vec{r}\,'|}$$

$\vec{r}$ = field point, $\vec{r}\,'$ = source point

Claimed general solution, for charge density ρ, is (next page)

$$\Phi(\vec{r}) = \int d^3r' \, \frac{\rho(\vec{r}\,')}{q} \, \phi_{\bullet}(\vec{r};\vec{r}\,')$$

To check, first apply ∇_r^2 to both sides (derivatives act only on field point):

$$\nabla^2\Phi(\vec{r}) = \int d^3r' \, \frac{\rho(\vec{r}\,')}{q} \, \underbrace{\nabla_r^2 \phi_{\bullet}(\vec{r};\vec{r}\,')}_{-q\frac{\delta^3(\vec{r}-\vec{r}\,')}{\epsilon_0}}$$

$$= -\frac{\rho(\vec{r})}{\epsilon_0} \checkmark$$

Next, note that the Φ defined by the integral has $\Phi = 0$ boundary condition at $|\vec{r}| \to \infty$, since $\phi_{\bullet}(\vec{r};\vec{r}\,') \longrightarrow 0$, $|\vec{r}| \longrightarrow \infty$ for any finite $|\vec{r}\,'|$.

It's not possible to have two (different) solutions of the Poisson equation for the same ρ and zero boundary condition at $|\bar{r}| = \infty$,

$$-\nabla^2\varphi_1 = \rho/\epsilon_0 \ , \ -\nabla^2\varphi_2 = \rho/\epsilon_0 \ ,$$

since then $-\nabla^2(\underbrace{\varphi_1 - \varphi_2}_{\varphi_3}) = 0$ would be another solution of Laplace equation, with zero boundary conditio at infinity, in addition to $\varphi_3 = 0$. (Solutions to the Laplace equation are unique.)

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3. We are given $\vec{V}$ and need to find f such that

$$\vec{V}' = \vec{V} + \vec{\nabla} f \quad \text{has zero divergence.}$$

$$0 = \vec{\nabla}\cdot\vec{V}' = \vec{\nabla}\cdot\vec{V} + \nabla^2 f$$

$$\Rightarrow -\nabla^2 f = \vec{\nabla}\cdot\vec{V} = ``\rho"/\epsilon_0$$

(It needn't have the interpretation of a charge density, but we can use the same symbol.)

$$\Rightarrow f(\vec{r}) = \int d^3r' \, \frac{\vec{\nabla}\cdot\vec{V}(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|}$$

Since $\vec{\nabla}\times\vec{V}' = \vec{\nabla}\times\vec{V} + \underbrace{\vec{\nabla}\times\vec{\nabla}f}_{0}$,

this shows it's always possible to modify a vector field,

$$\vec{V} \longrightarrow \vec{V}'$$

without changing its curl, so that it has zero divergence.