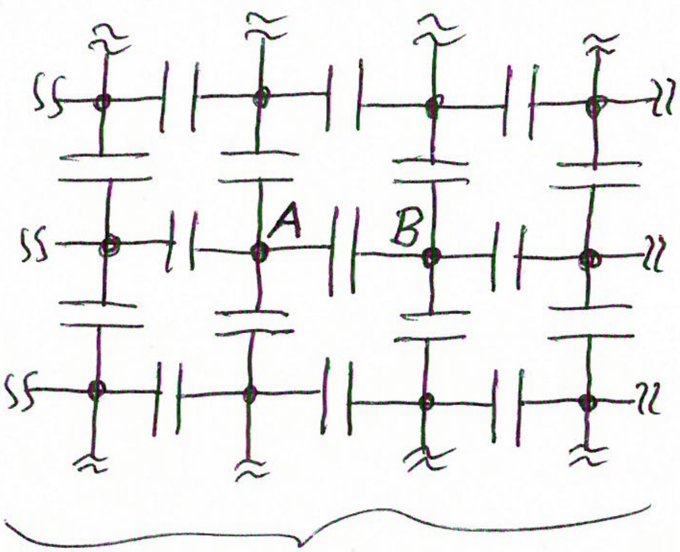
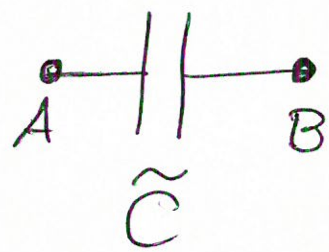
Assignment 7 Solutions

infinite square grid of capacitors

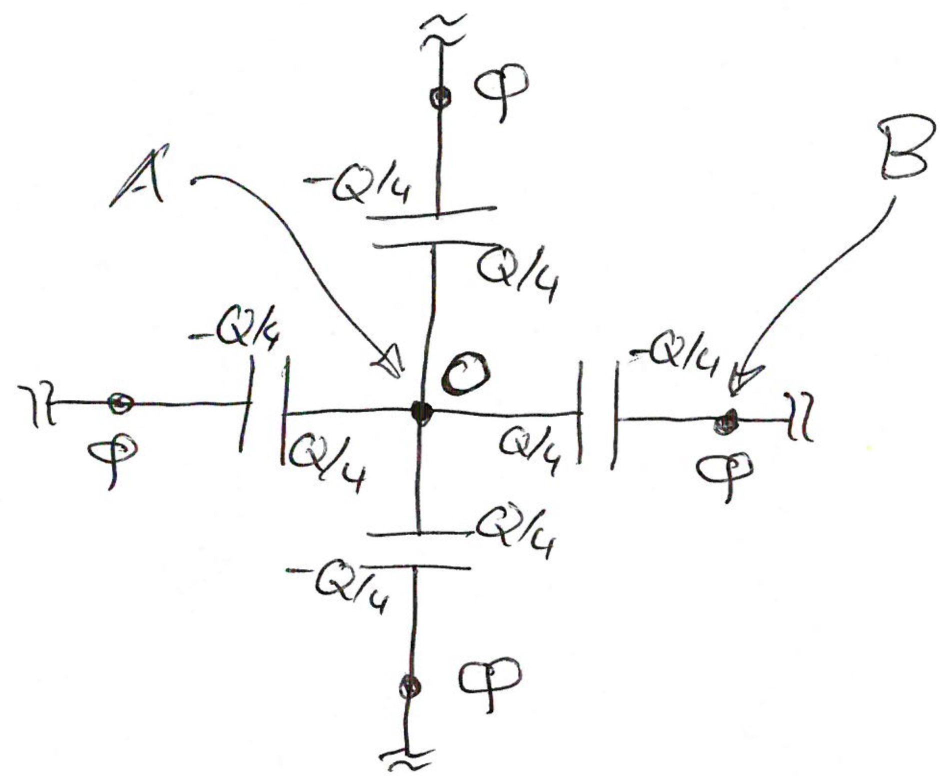




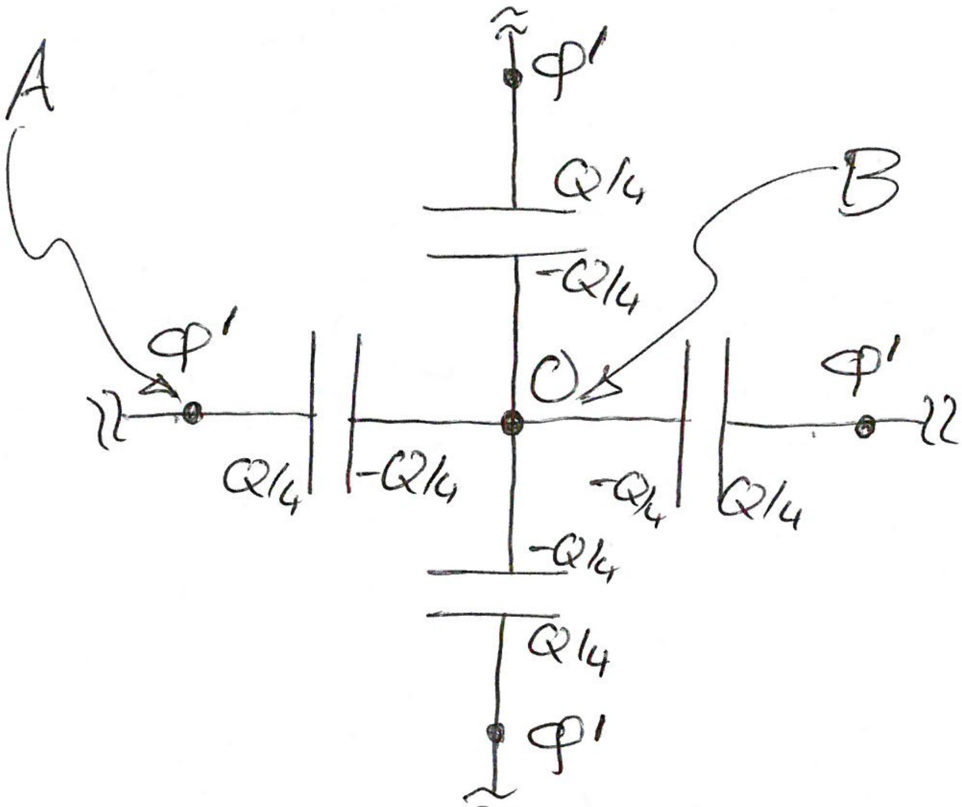
all capacitors have
$C = 1F$

If $+Q$ is placed on node A, and $-Q$ on node B, and $\varphi_A - \varphi_B = V$, then the infinite grid behaves like a single capacitor $\tilde{C}$ with capacitance Q/V. We will work out this ratio using symmetry.

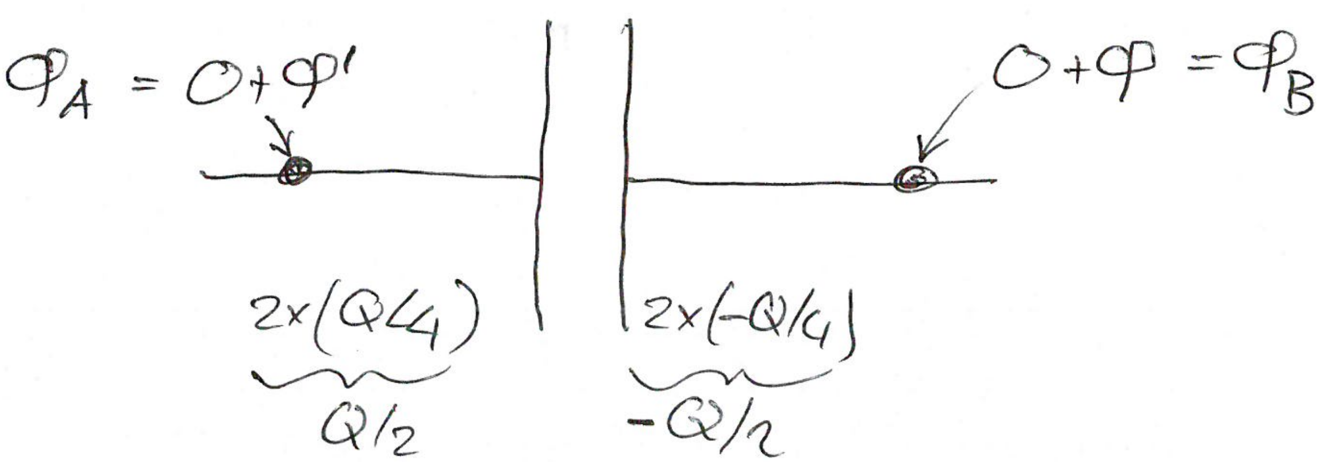
When just $+Q$ is placed at A, and we set $\varphi_A = 0$, we get the following charge/potential arrangement near A:



Where $\varphi = \dfrac{-Q/4}{C}$ on the four nodes adjacent to A. Now, make a similar drawing symmetric about B, but with charge $-Q$ at that node (next page).



$\varphi' = \frac{+Q/4}{C}$. Now superpose the two drawings, focusing on nodes A, B and the charge on the capacitor between them:



In the superposition, charge $+Q$ is placed on node A, and charge $-Q$ is placed on node B, as asked in the problem. ④

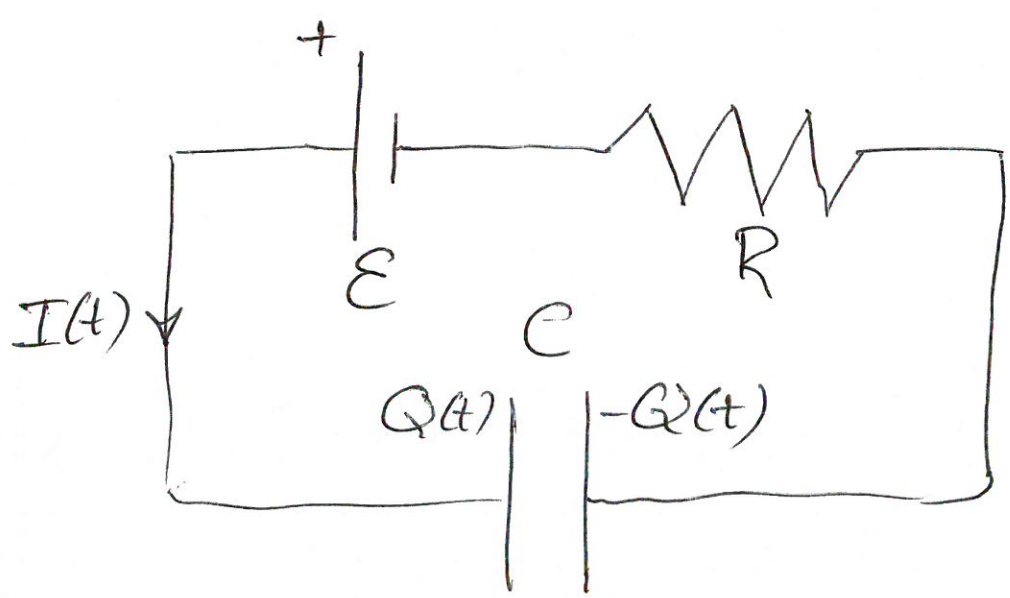
$$V = \varphi_A - \varphi_B = \varphi' - \varphi = \frac{Q}{4C} - \left(\frac{-Q}{4C}\right)$$

$$= \frac{Q}{2C}$$

From this we see that

$$\tilde{C} = Q/V = 2C = 2F.$$

Energy/power in simple RC circuit



At $t=0$ $Q=0$, so the potential across the capacitor is $0/C = 0$, the loop rule simplifies to

$$\mathcal{E} - I(0)R = 0$$

$$\Rightarrow I(0) = \mathcal{E}/R$$

At $t=\infty$ $I=0$, so there is no potential drop at the resistor, the loop rule again simplifies:

$$\mathcal{E} - \frac{Q(\infty)}{C} = 0$$

$$\Rightarrow Q(\infty) = \mathcal{E}C$$

From lecture, $Q(t) = \mathcal{E}C(1-e^{-t/\tau})$

$$\tau = RC$$

power provided by EMF: ⑥

$$\mathcal{E} I(t) = \mathcal{E}\dot{Q} = \mathcal{E}\frac{\mathcal{E}C}{\tau} e^{-t/\tau}$$

$$= \frac{\mathcal{E}^2}{R} e^{-t/\tau}$$

power consumed by (stored in) the capacitor:

$$\frac{d}{dt}\left(\frac{1}{2}\frac{Q^2(t)}{C}\right) = \frac{Q}{C}\dot{Q} = \mathcal{E}(1-e^{-t/\tau})\frac{\mathcal{E}}{R}e^{-t/\tau}$$

$$= \frac{\mathcal{E}^2}{R}\left(e^{-t/\tau} - e^{-2t/\tau}\right)$$

power dissipated by resistor:

$$RI^2 = \frac{\mathcal{E}^2}{R} e^{-2t/\tau}$$

$$\frac{\mathcal{E}^2}{R}\Big(\underbrace{+e^{-t/\tau}}_{\text{EMF}} - \underbrace{\left(e^{-t/\tau} - e^{-2t/\tau}\right)}_{\text{capacitor}} - \underbrace{e^{-2t/\tau}}_{\text{resistor}}\Big) = 0$$

total energy provided by EMF:

$$\int_0^\infty \frac{\mathcal{E}^2}{R} e^{-t/\tau}\, dt = \frac{\mathcal{E}^2}{R}(-\tau) e^{-t/\tau}\Big|_0^\infty$$

$$= \frac{\mathcal{E}^2}{R}\tau = \mathcal{E}^2 C$$

final energy stored in capacitor:

$$\frac{1}{2} C\, V(\infty)^2 = \frac{1}{2} C \mathcal{E}^2$$

From this we see that energy

$$\mathcal{E}^2 C - \frac{1}{2}\mathcal{E}^2 C = \frac{1}{2}\mathcal{E}^2 C$$

was dissipated (lost to heat) in the resistor.