What is the result of combining rotations about different axes?

In a "compound boost" we make a Lorentz transformation from frame 1 to frame 2, with some relative velocity, and follow that with a Lorentz transformation from frame 2 to frame 3, with some other velocity. You were told in lecture that frames 1 and 3 are not just related by a single boost — for the sum of the relative velocities — in that there is also a *rotation* between frames 1 and 3. This is a strange fact, and this short note is meant to make it less mysterious by giving you an example of a closely related effect involving just rotations.

In mechanics you learned that angular velocities indeed add as vectors. For example, if you are on a merry-go-round with angular velocity $\boldsymbol{\omega}_{12}$ relative to the earth, and spin a bicycle wheel with angular velocity $\boldsymbol{\omega}_{23}$ relative to the merry-go-round, then the angular velocity of the wheel relative to the earth is $\boldsymbol{\omega}_{12} + \boldsymbol{\omega}_{23}$. But angular velocity corresponds to an infinitesimal rotation, and we should not be surprised to get a different result when we combine finite rotations.

Here is a simple example showing the failure of "addition of small rotations". Positive rotations (right-hand-rule) by angle θ about the x-axis are defined by the matrix

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{\theta^2}{2} & -\theta \\ 0 & \theta & 1-\frac{\theta^2}{2} \end{pmatrix},$$

where the matrix on the right is the approximation that keeps terms up to second order in the small angle θ. For small rotations about the y-axis the corresponding matrices are

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \approx \begin{pmatrix} 1-\frac{\theta^2}{2} & 0 & \theta \\ 0 & 1 & 0 \\ -\theta & 0 & 1-\frac{\theta^2}{2} \end{pmatrix}.$$

Now, if small rotations added perfectly, then the "compound rotation" $R_y(\theta)R_x(\theta)$ — first about $\hat{\mathbf{x}}$ and then about $\hat{\mathbf{y}}$ — would be a rotation about $\hat{\mathbf{n}} = (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$ by amount $\sqrt{2}\,\theta$. By rotating $\hat{\mathbf{n}}$ to $\hat{\mathbf{x}}$, with a z-axis rotation by $-\pi/4$, performing the rotation $R_x(\sqrt{2}\,\theta)$ and rotating $\hat{\mathbf{x}}$ back to $\hat{\mathbf{n}}$ with a z rotation by $+\pi/4$, we get the matrix for general rotations about $\hat{\mathbf{n}}$:

$$R_n(\sqrt{2}\,\theta) = R_z(\pi/4)\, R_x(\sqrt{2}\,\theta)\, R_z(-\pi/4).$$

Since

$$R_z(\pm\pi/4) = \begin{pmatrix} 1/\sqrt{2} & \mp 1/\sqrt{2} & 0 \\ \pm 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

this rotation matrix works out to be

$$\begin{aligned} R_n(\sqrt{2}\,\theta) &= \begin{pmatrix} \frac{1}{2}\cos\left(\sqrt{2}\theta\right)+\frac{1}{2} & \frac{1}{2}-\frac{1}{2}\cos\left(\sqrt{2}\theta\right) & \sin\left(\sqrt{2}\theta\right)/\sqrt{2} \\ \frac{1}{2}-\frac{1}{2}\cos\left(\sqrt{2}\theta\right) & \frac{1}{2}\cos\left(\sqrt{2}\theta\right)+\frac{1}{2} & -\sin\left(\sqrt{2}\theta\right)/\sqrt{2} \\ -\sin\left(\sqrt{2}\theta\right)/\sqrt{2} & \sin\left(\sqrt{2}\theta\right)/\sqrt{2} & \cos\left(\sqrt{2}\theta\right) \end{pmatrix} \\ &\approx \begin{pmatrix} 1-\frac{\theta^2}{2} & \frac{\theta^2}{2} & \theta \\ \frac{\theta^2}{2} & 1-\frac{\theta^2}{2} & -\theta \\ -\theta & \theta & 1-\theta^2 \end{pmatrix}. \end{aligned}$$

If small rotations combined naively, then the inverse of this applied to the compound rotation $R_y(\theta)R_x(\theta)$ should give the identity transformation. Instead, for small θ we obtain

$$R_n(-\sqrt{2}\,\theta)R_y(\theta)R_x(\theta) \approx \begin{pmatrix} 1 & \frac{\theta^2}{2} & 0 \\ -\frac{\theta^2}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is a rotation about the z-axis by angle $-\theta^2/2$. When the compound rotation is generated in the opposite order,

$$R_n(-\sqrt{2}\,\theta)R_x(\theta)R_y(\theta) \approx \begin{pmatrix} 1 & -\frac{\theta^2}{2} & 0 \\ \frac{\theta^2}{2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

the rotation about the z-axis is by angle $+\theta^2/2$.

Notice that the correction to the naive addition-of-angular-velocity-rotation (about $\hat{\mathbf{n}}$) is second order, because the rotations about x and y were each by θ. The frame rotation effect is similar: the resulting rotation, about the axis perpendicular to both boosts, has angle $(1/2)uv/c^2$. Even the factor of $1/2$ turns out to be the same!