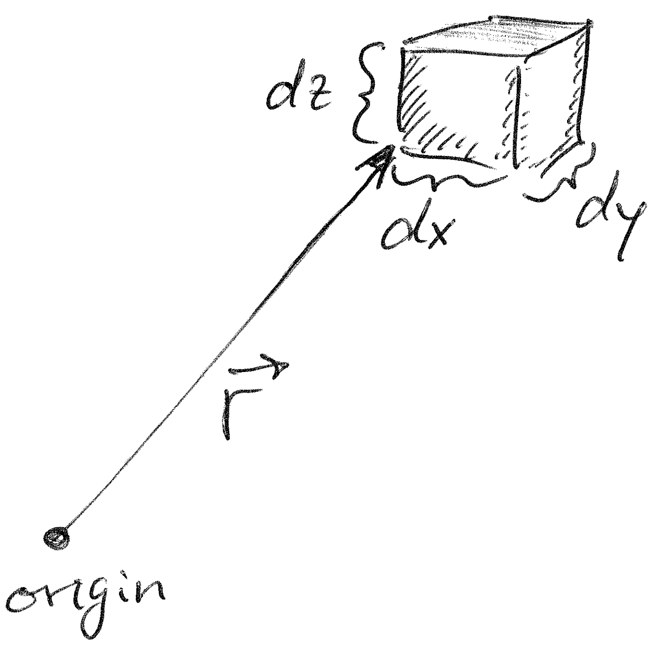
Lecture 8

Consider a fixed collection of charges and the electric field $\vec{E}$ they produce.
A convenient description of large number of charges is the <u>charge density distribution</u> $\rho(\vec{r})$:

$$\rho(\vec{r})\, d^3r = \left(\begin{array}{l}\text{total charge within the} \\ \text{volume element } d^3r \\ \text{at } \vec{r}\end{array}\right)$$

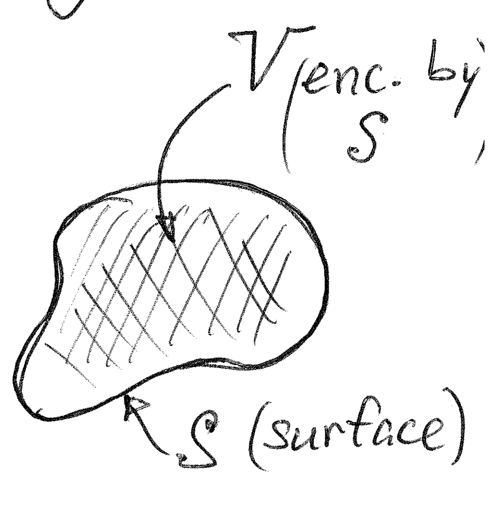


$$d^3r = dx\,dy\,dz$$

We will see later that $\rho(\vec{r})$ can even describe isolated point charges. For now, you should think of it as the appropriate description when the separation of the individual charges is <u>small</u> on the scale of interest, when charge is in effect a continuous quantity defined everywhere in space.

Gauss's law refers to the net charge enclosed by a surface S. In terms of $\rho(\vec{r})$ the enclosed charge takes the following form:

$$\int_{\underbrace{\text{Vol. enc. by } S}_{V}} d^3r\, \rho(\vec{r}) = Q_{enc.}$$



Both sides of Gauss's law are now expressed as integrals:

$$\underbrace{\oint_{\text{boundary}(V)}}_{S} \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}\int_V \rho(\vec{r})\,d^3r$$

We saw that the "divergence theorem" (lecture 7) allows us to relate a flux integral over the boundary of a volume to a volume integral. Applying that general theorem to the left-hand-side of Gauss's law above,

$$\oint_{\text{boundary}(V)} \vec{E}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{E}\, d^3r \quad \text{(div. thm.)}$$

Gauss

$$= \frac{1}{\epsilon_0}\int_V \rho(\vec{r})\, d^3r$$

Gauss's law asserts the two volume integrals are equal for <u>any choice of volume V</u> (equivalent to the freedom of choosing the boundary surface S). This extends to arbitrarily small volumes V; taking V to be a single volume element d^3r at $\vec{r}$ we find:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

Both sides are to be evaluated at the same point $\vec{r}$, but this can be any point. If we have a region that is free of charge, so $\rho(\vec{r})=0$, then $\vec{\nabla}\cdot\vec{E}(\vec{r})=0$ in that same region (the electric field is "divergence-free" where there is no charge density). The differential equation relating $\vec{E}$ and ρ is called the "differential form" of Gauss's law.

It is recognized as a more fundamental statement, than the flux integral form, because it relates <u>local</u> quantities.

Can we still use the differential form of Gauss's law in the context of a simple point charge q?

It should be possible, because our $\rho(\vec{r})$ could describe a very <u>compact</u> distribution of charges (e.g. a nucleus) so that at the scale of interest it is completely characterized by its position $\vec{r}_0$ and net charge, q.

Except for points $\vec{r}$ close to $\vec{r}_0$ the electric field produced by such a compact $\rho(\vec{r})$ is that of a point charge q located at $\vec{r}_0$:

$$\vec{E}(\vec{r}) = K \frac{q}{|\vec{r}-\vec{r}_0|^3}(\vec{r}-\vec{r}_0) \quad (\vec{r} \neq \vec{r}_0)$$

By the differential form of Gauss's law, this $\vec{E}$ is divergence free at all $\vec{r}$ away from $\vec{r}_0$, and the divergence we get near $\vec{r}_0$ is such that its integral equals the net charge q divided by ϵ_0. The physicist Dirac came up with a useful notation for a distribution that is zero except near a particular point such that it has a particular integral:

(1) $\delta^3(\vec{r}-\vec{r}_0) = 0$ at all $\vec{r} \neq \vec{r}_0$

(2) $\int \delta^3(\vec{r}-\vec{r}_0)\, d^3r = 1$ (unit integral)

Properties (1) and (2) define the

"Dirac delta function" at point $\vec{r}_0$ in 3D space [It's better to think of it as a "distribution" than an actual "function". A distribution can be defined more abstractly in terms of its properties, as we have done. The term "function" has persisted in the physics literature, and as a result δ^3 is usually defined as the limit of a sequence of functions that become ever narrower in space and larger in value (to keep the integral constant). However, this is a distraction since it is only the integral of $\delta^3(\vec{r}-\vec{r}_0)$ and the location of its nonzero part that matters, ~~and not~~]

The divergence of the point-charge electric field, being $1/\epsilon_0$ times the charge density of an arbitrarily compact distribution, is therefore ~~therefore~~ therefore expressible in terms of the Dirac delta:

$$\vec{\nabla} \cdot \left(Kq \frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \right) = \frac{1}{\epsilon_0} \underbrace{q\,\delta^3(\vec{r}-\vec{r}_0)}_{\substack{\rho(\vec{r}) \text{ with} \\ \text{integral } q}}$$

Divide by Kq and use $\epsilon_0 = \frac{1}{4\pi K}$:

$$\vec{\nabla} \cdot \left(\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3} \right) = 4\pi\,\delta^3(\vec{r}-\vec{r}_0)$$

special case, $\vec{r}_0 = 0$: $\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi\,\delta^3(\vec{r})$

Q : Make sense of the <u>units</u> of the last equation

Regardless of the divergence of the electric field (whether or not there is charge), the electric field in electro<u>statics</u> always defines a conservative force field. Suppose we have a fixed distribution of charge that produces a fixed, static field $\vec{E}(\vec{r})$. A "test charge" q_0 will experience a conservative force in this field and we can define a potential energy function for it as it is moved from place to place.

$$U_0(\vec{r}) = \begin{pmatrix} \text{potential energy of} \\ \text{test charge } q_0 \end{pmatrix}$$

$$U_0(\vec{r}) - U_0(\vec{r}_0) = -\int_{\substack{\text{path from} \\ \vec{r}_0 \text{ to } \vec{r}}} q_0 \vec{E}(\vec{r}\,') \cdot d\vec{r}$$

Up to the arbitrariness of the energy $U_0(\vec{r}_0)$ at the reference point $\vec{r}_0$, this defines a unique scalar function $U_0(\vec{r})$ given an arbitrary field $\vec{E}(\vec{r})$. Since U_0 is proportional to q_0, we divide it by q_0 and thereby make it only depend on $\vec{E}$:

$$U_0(\vec{r})/q_0 \equiv \varphi(\vec{r})$$

$$= \left(\text{the "electric potential" at point } \vec{r} \right)$$

$$\varphi(\vec{r}) - \varphi(\vec{r}_0) = -\int_{\vec{r}_0 \text{ to } \vec{r}} \vec{E} \cdot d\vec{r}\,'$$

In addition to its energy interpretation, the function $\varphi(\vec{r})$ is useful because it provides an equivalent and in many ways simpler (scalar!) representation of the information in $\vec{E}$. The formula above shows how we get φ from $\vec{E}$ (up to an additive constant). We will now see that we can also go the other way, i.e. recover $\vec{E}$ from φ:

$$\varphi(\vec{r}+\hat{x}dx) - \varphi(\vec{r}_0) = \int_{\vec{r}_0 \text{ to } \vec{r}+\hat{x}dx} \cdots$$

$$\varphi(\vec{r}) - \varphi(\vec{r}_0) = \int_{\vec{r}_0 \text{ to } \vec{r}} \cdots$$

Subtract left sides:

$$\varphi(\vec{r}+\hat{x}dx) - \varphi(\vec{r}) = \left.\frac{\partial\varphi}{\partial x}\right|_{\vec{r}} \cdot dx$$

Subtract right sides:

$$-\int_{\vec{r} \text{ to } \vec{r}+\hat{x}dx} \vec{E}(\vec{r}\,')\cdot d\vec{r}\,' = -E_x(\vec{r})\,dx$$

Of course we get the same formulas when we displace the endpoint $\vec{r}$ by $\hat{y}\,dy$ and $\hat{z}\,dz$. All three are expressed by the vector formula

$$\vec{\nabla}\varphi = -\vec{E}$$

where both sides are to be evaluated at the same, arbitrary point $\vec{r}$.

From $\varphi = U_0/q_0$ we see that electric potential has units J/C. ~~From the gradient formula above we~~ This combination is called the "volt" or V.