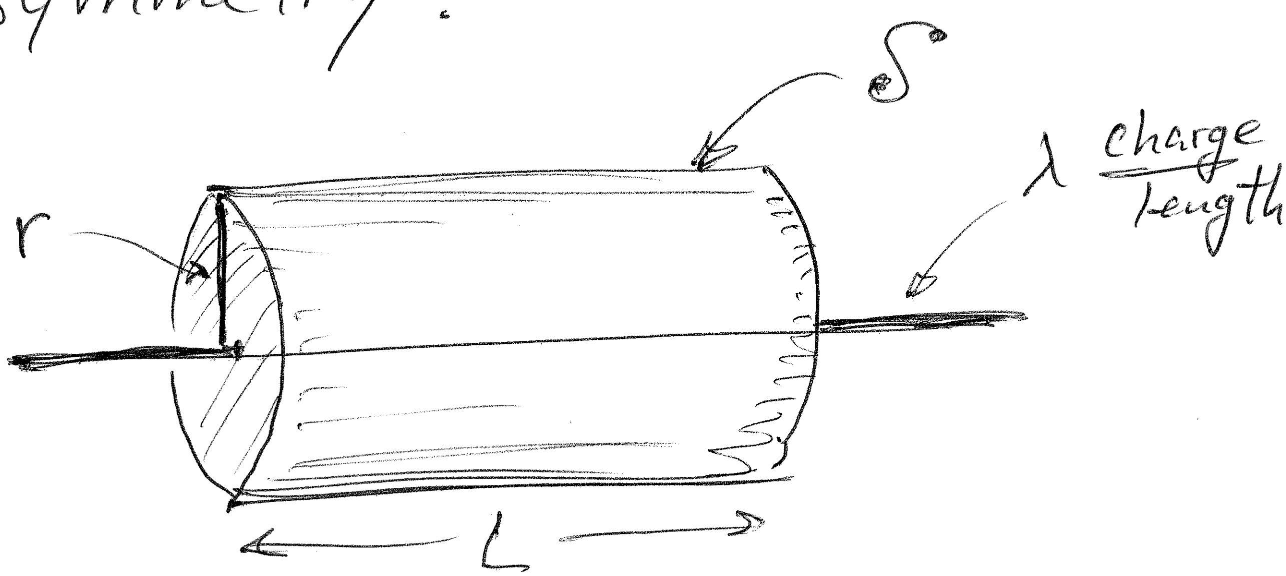
Lecture 7

Having laid the symmetry groundwork for the electric field of an infinite line charge in lecture 6, the Gauss's law finale is easy:

$$\vec{E} = E(r)\,\hat{r}$$ (radial; scalar magnitude only dependent on r)

choose S with cylindrical symmetry:



$\vec{E} \cdot d\vec{a} = 0$ on cylinder ends

$\vec{E} \cdot d\vec{a} = E(r)\, |d\vec{a}|$ on curved part of surface

$$\oint_S \vec{E} \cdot d\vec{a} = E(r) \int_{\substack{\text{curved} \\ \text{surface}}} |d\vec{a}| = E(r)\, 2\pi r L$$

Gauss's law :

$$E(r)\, 2\pi r L = \frac{Q_{enc}}{\epsilon_0} = \frac{\lambda \cdot L}{\epsilon_0}$$

$$\Rightarrow \quad E(r) = \frac{1}{2\pi\epsilon_0} \frac{\lambda}{r}$$

The $1/r$ decay of $\vec{E}$ with distance from a line charge is also the decay law if ~~the~~ "space" was a 2D plane and Gauss's law was appropriately modified.

Gauss's law in 2D "world":

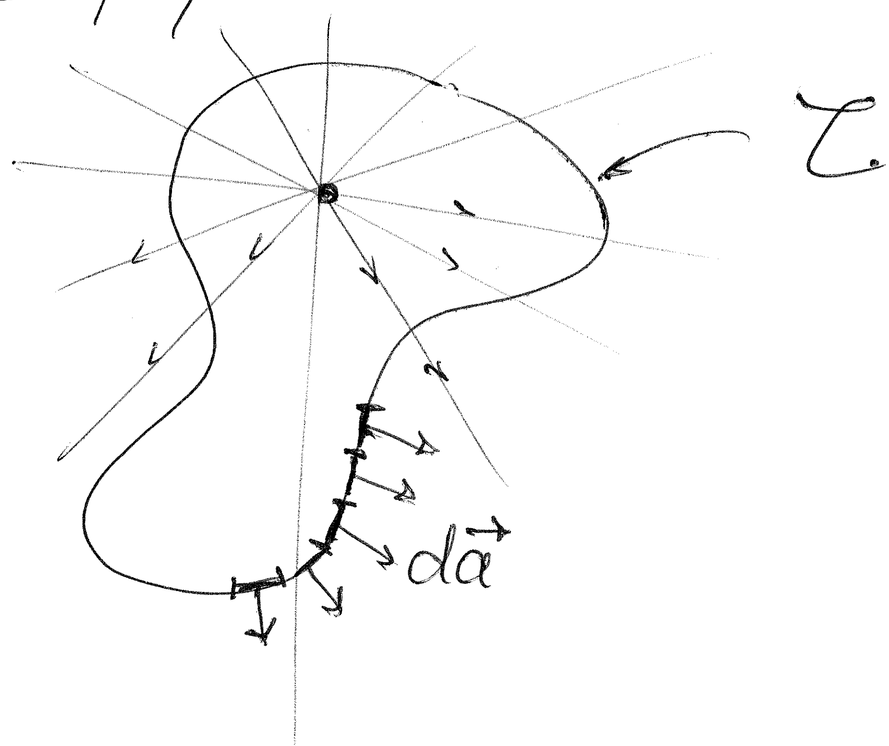
$$\oint_{\mathcal{C}} \vec{E} \cdot d\vec{a} = \frac{Q_{enc}}{\epsilon'}$$

$\mathcal{C}$ = closed curve (1D)

Q_{enc} = charge enclosed by $\mathcal{C}$

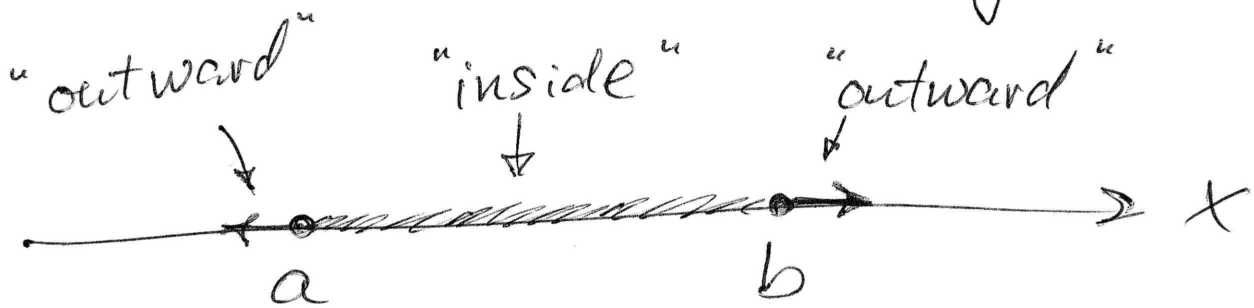
$d\vec{a}$ = <u>line</u> element of $\mathcal{C}$

ϵ' = physical constant



Our next goal is to express the content of Gauss's law in <u>local</u> terms. As a first step we need to recall a general relationship between particular kinds of integrals in D-dimensions and (D-1)-dimensions:

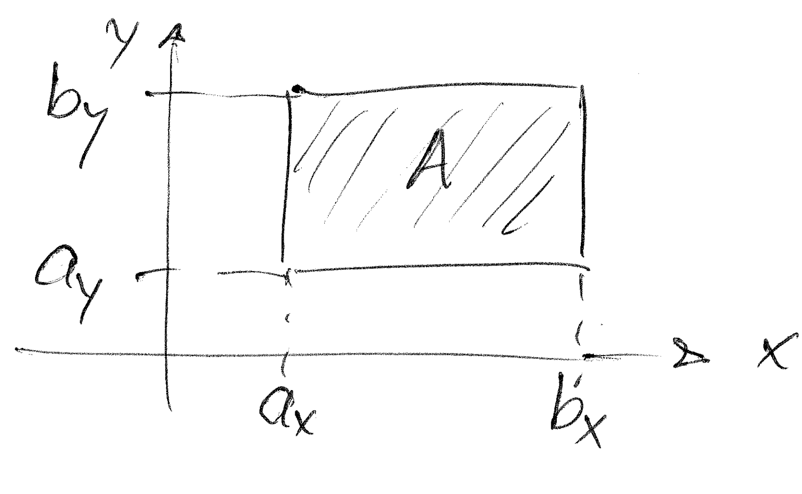
$$\underbrace{\int_a^b \frac{df}{dx}\,dx}_{\substack{\text{D=1 integral} \\ \text{of a derivative}}} = \underbrace{f(b) - f(a)}_{\substack{\text{D=0 "integral"} \\ \text{associated with} \\ \text{boundary of D=1} \\ \text{integral}}}$$



f is actually a vector field, although in 1 dimension there is only 1 component so we don't need to express this with the notation. Let's see how things generalize in one higher dimension:

$$\int_A \left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} \right) dx\, dy = ?$$

This D=2 integral is performed inside a 2D region A. To keep things simple, let A be a rectangle



In our 2D integral we can trivially "do" <u>one</u> of the two integrals with the following result :

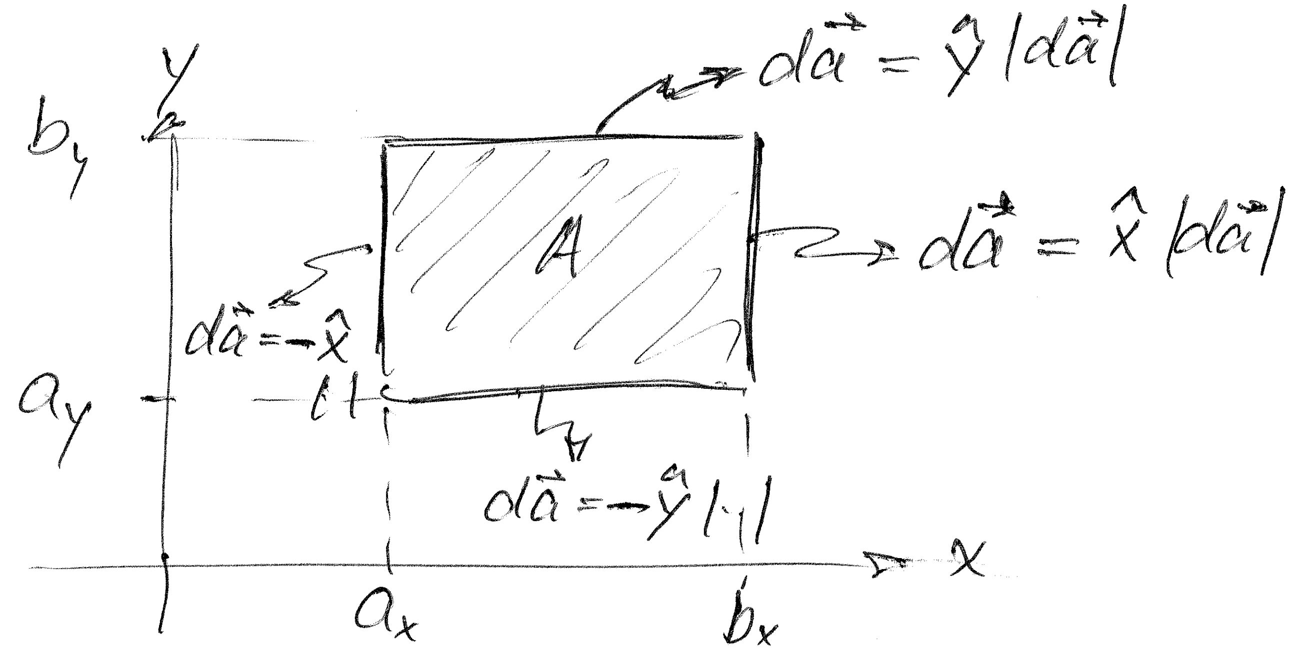
$$(\text{2D integral}) = \int_{a_y}^{b_y} f_x(b_x, y)\, dy$$

$$- \int_{a_y}^{b_y} f_x(a_x, y)\, dy + \int_{a_x}^{b_x} f_y(x, b_y)\, dx$$

$$- \int_{a_x}^{b_x} f_y(x, a_y)\, dx$$

$$= \int_{\text{boundary}(A)} (f_x \hat{x} + f_y \hat{y}) \cdot d\vec{a}$$

This is a correct statement if we can define a vector element $d\vec{a}$ on the 1D boundary of A. Well, consider this:



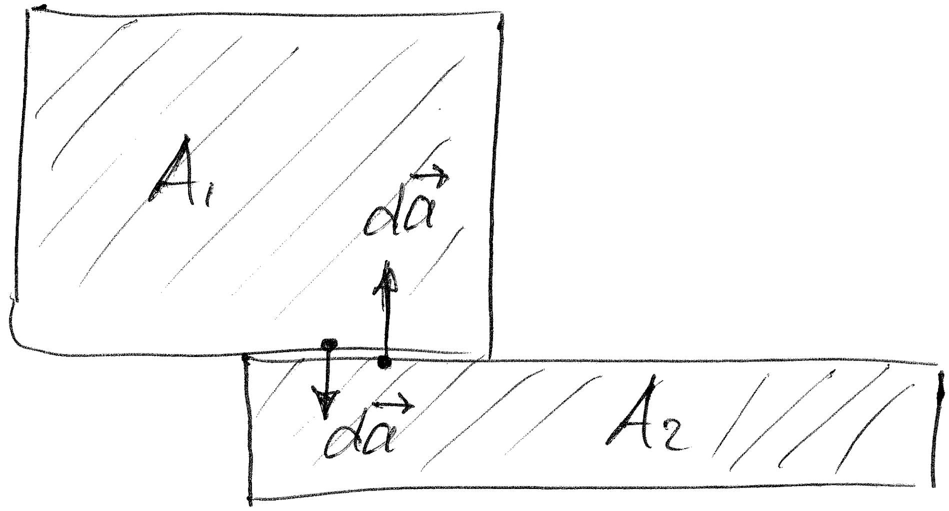
These definitions — complete with signs — correctly reproduce our 4 boundary integrals. ($|d\vec{a}| = dx$ or dy, whichever direction we're integrating.) But these $d\vec{a}$'s are exactly the <u>outward</u> <u>normal</u> surface/boundary elements.

Summarizing, in more compact notation,

$$\int_A (\vec{\nabla} \cdot \vec{f}) \, dx\,dy = \oint_{\text{boundary}(A)} \vec{f} \cdot d\vec{a}$$

where $\vec{\nabla} \cdot \vec{f} = \dfrac{\partial f_x}{\partial x} + \dfrac{\partial f_y}{\partial y}$ is the "divergence".

This generalizes in the obvious way to any number of dimensions. We can also see why it works for more complex regions A by combining rectangular ones:



$$\oint_{\text{bound}(A_1)} \vec{f}\cdot d\vec{a} + \oint_{\text{bound}(A_2)} \vec{f}\cdot d\vec{a} =$$

$$\oint_{\text{bound}(A)} \vec{f}\cdot d\vec{a}$$

(integrals over common boundary cancel because $d\vec{a}$'s have opposite sign)