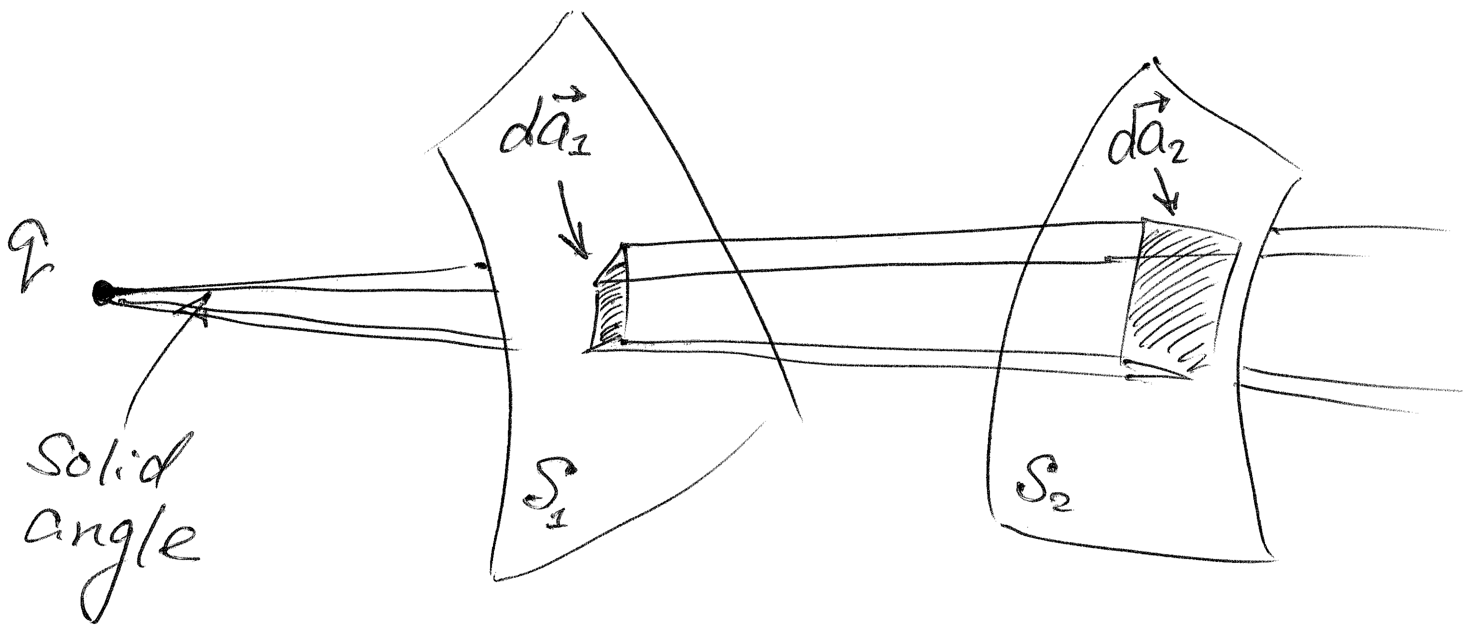
Lecture 5

Consider a point charge q and that part of its electric field subtended by a small solid angle:

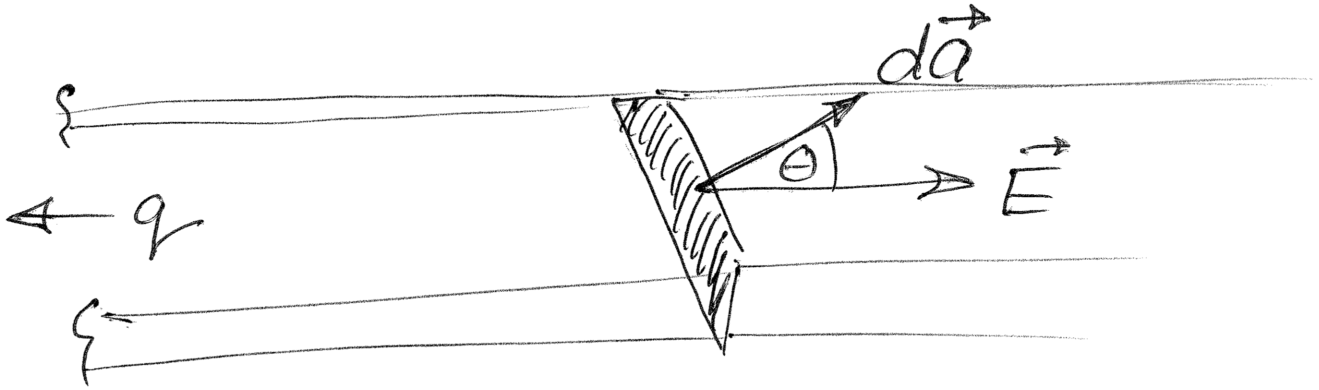


The solid angle intercepts the two surfaces S_1 and S_2 at two surface elements $d\vec{a}_1$ and $d\vec{a}_2$. We will shortly show that, up to a sign,

$$d\vec{a}_1 \cdot \vec{E}_1 = d\vec{a}_2 \cdot \vec{E}_2$$

where $\vec{E}_1$ and $\vec{E}_2$ are the electric fields evaluated at the two surface elements ($\vec{E}$ is nearly constant over each element when the solid angle is very small). The question of sign hinges upon what we define to be "inside" or "outside" for each surface. We will address this later.

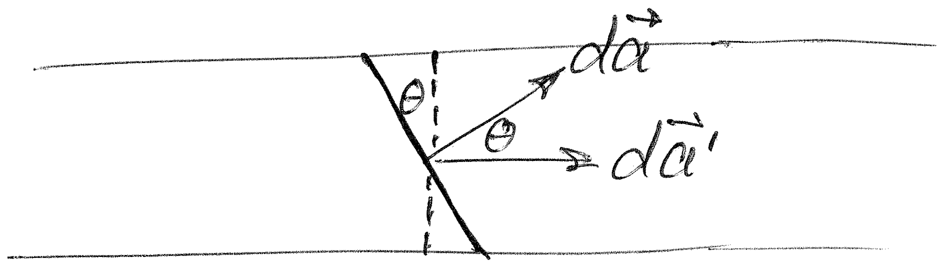
Suppose a surface element $d\vec{a}$ has angle θ with respect to the radial electric field $\vec{E}$:



Compare $|d\vec{a}|$ to the area of

a surface element $d\vec{a}'$
parallel to $\vec{E}$:

2D side view:



$|d\vec{a}| \propto$ hypotenuse
$|d\vec{a}'| \propto$ vertical line

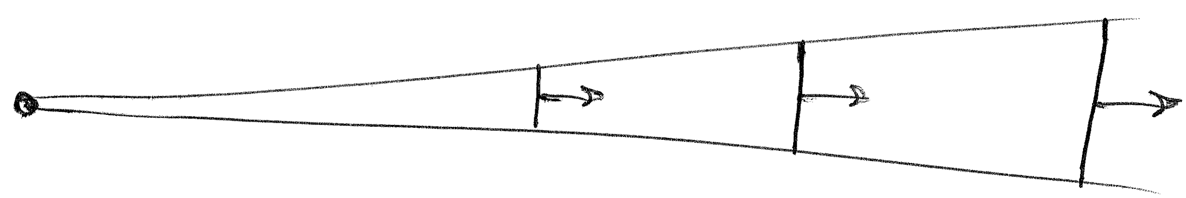
$$|d\vec{a}| \cos\theta = |d\vec{a}'|$$

$$\vec{E} \cdot d\vec{a} = \cos\theta \, |\vec{E}| \, |d\vec{a}|$$

$$= |\vec{E}| \, |d\vec{a}'|$$

$$= \vec{E} \cdot d\vec{a}'$$

So we get the same flux $d\Phi$ for any angle between $\vec{E}$ and $d\vec{a}$. The flux $d\Phi$ is also unchanged when $d\vec{a}$ is moved radially; this is immediately clear when we compare elements $d\vec{a}$ all parallel to $\vec{E}$:

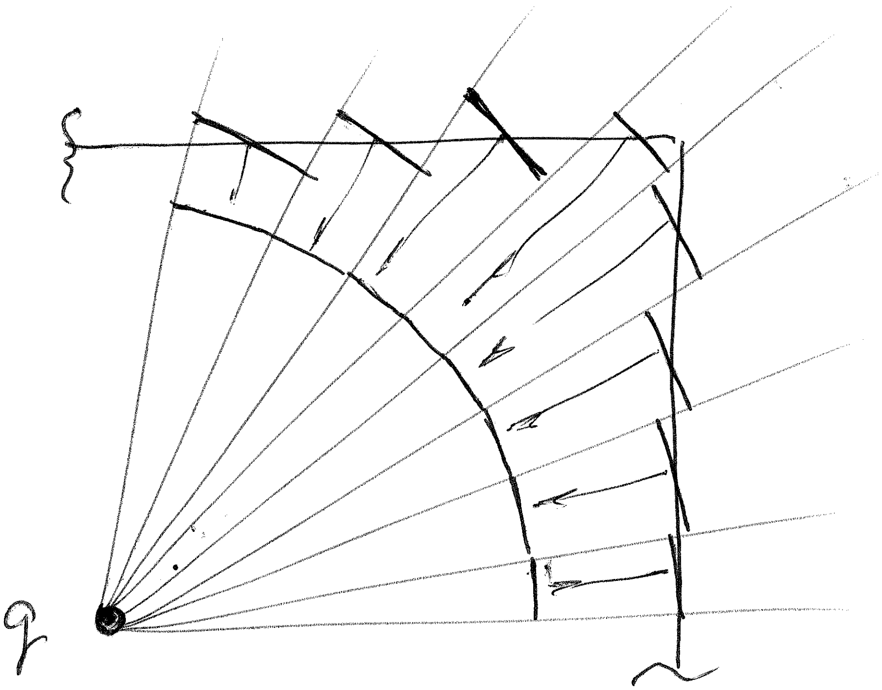
q 

r = distance between q and $d\vec{a}$

$$|\vec{E}| \propto 1/r^2 \quad , \quad |d\vec{a}| \propto r^2$$

$$\Rightarrow \quad \vec{E}\cdot d\vec{a} = \text{independent of } r$$

Suppose the surface $\mathcal{S}$ in Gauss's law is a <u>box</u>. Here's how to transform the box into a sphere without changing the flux $\underline{\Phi}$:



first rotate $d\vec{a}$'s parallel to $\vec{E}$, then move radially

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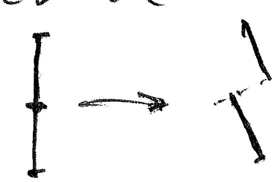
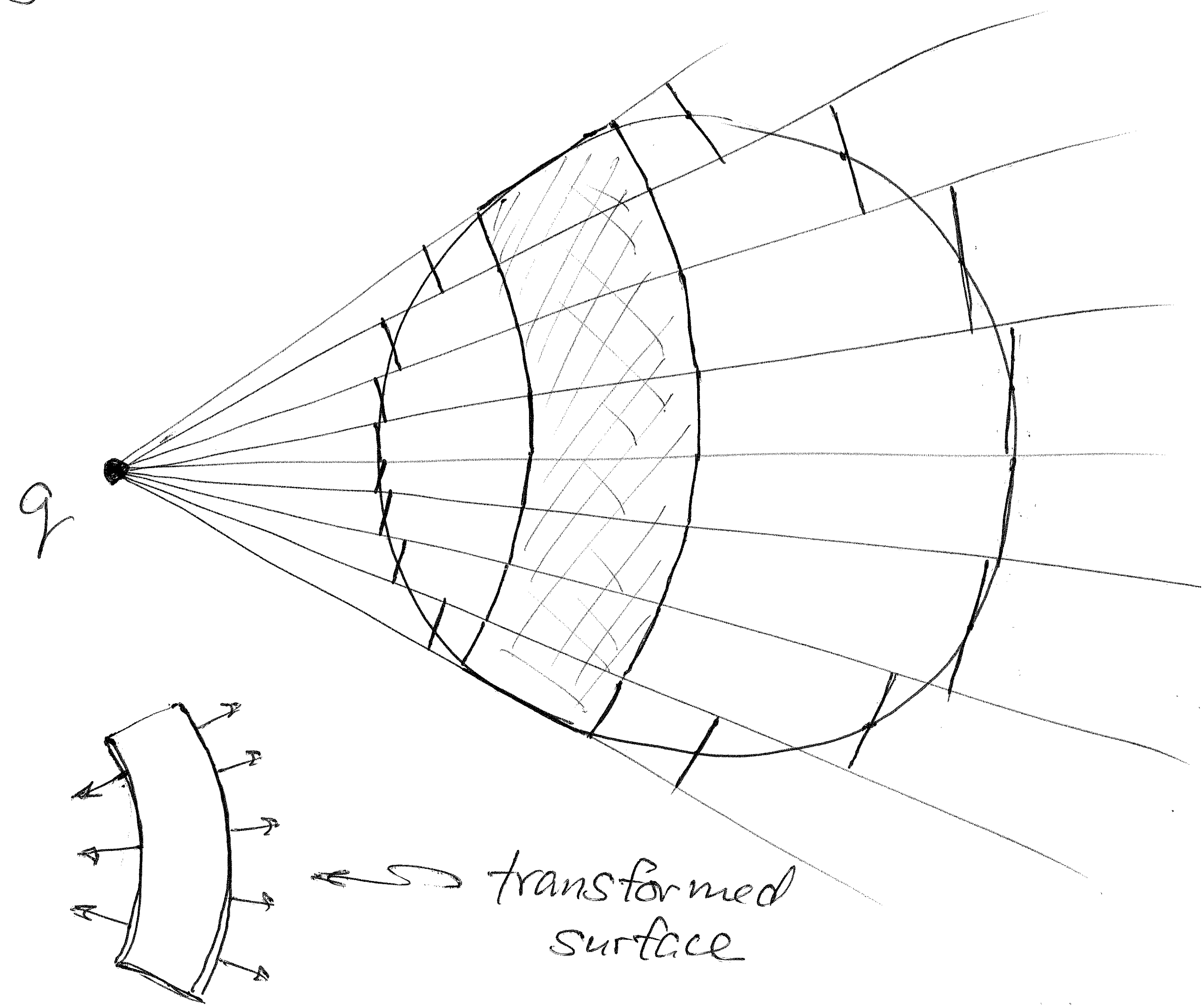
The <u>gaps</u> that open up when elements are rotated



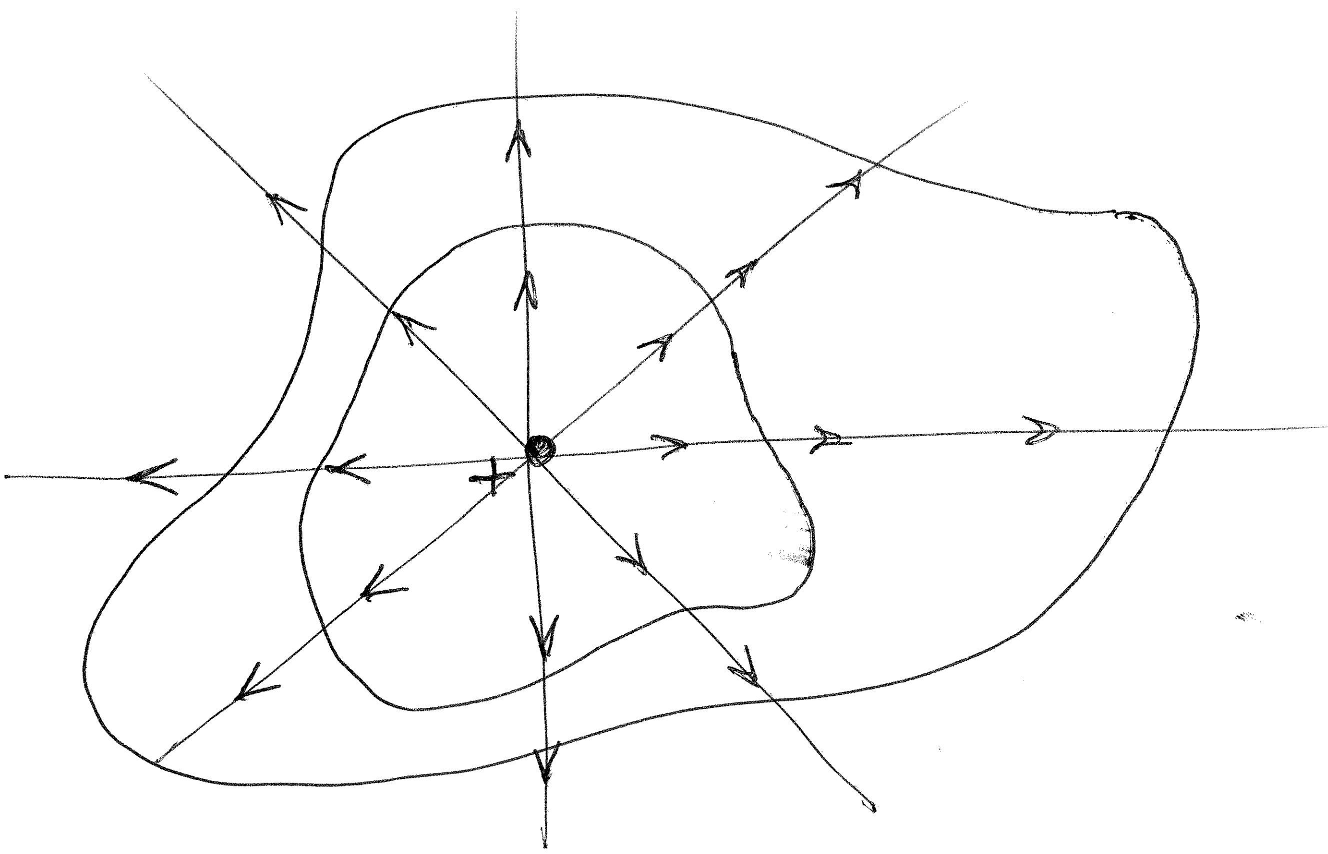
are not a problem because, by construction, they represent surface elements perpendicular to $\vec{E}$ with zero flux.

Now consider a charge q outside a spherical surface. Here's a transformation that shows the flux is zero:

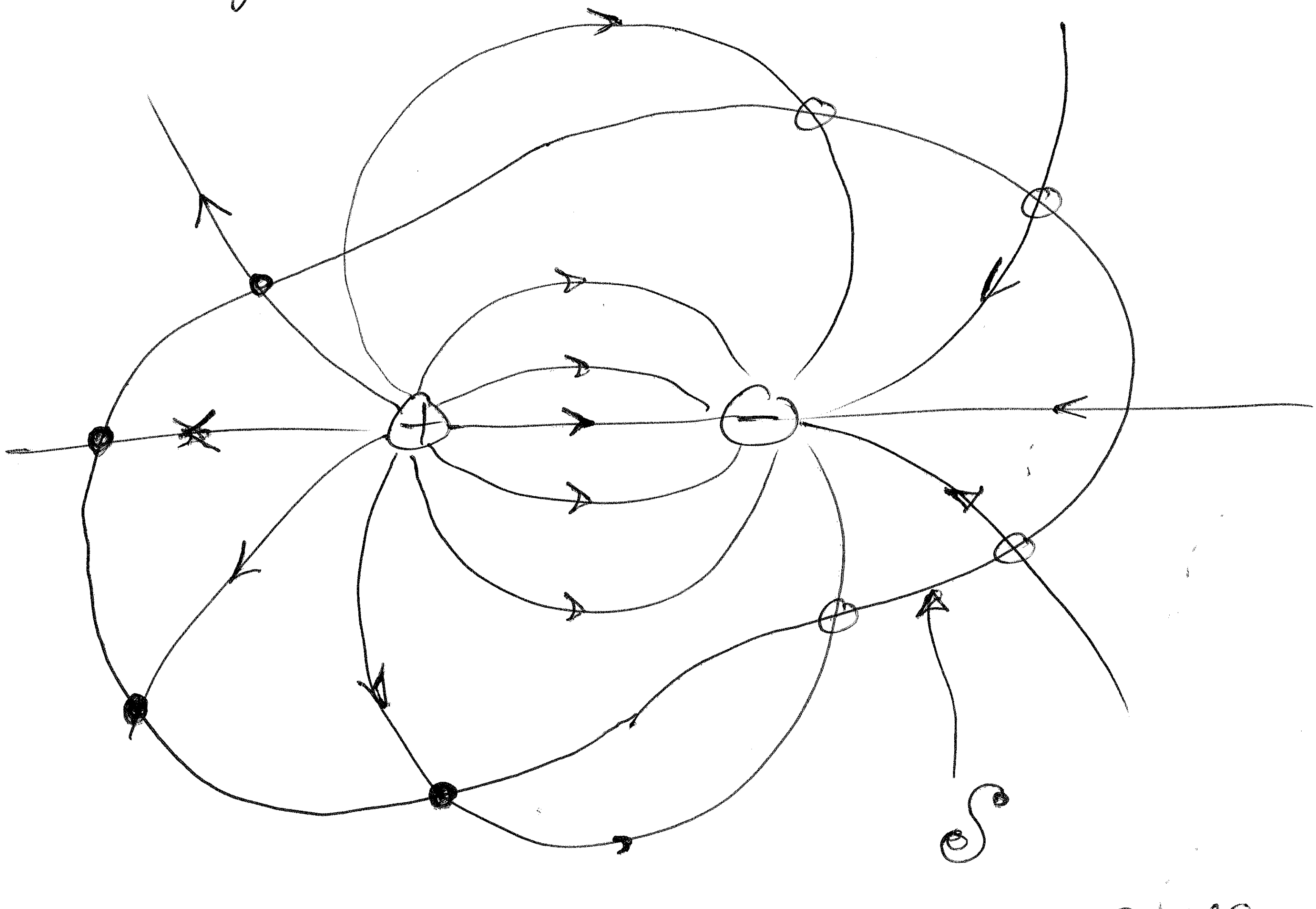


The transformed surface comprises two spherical caps joined by a section of a cone that has zero flux ($\vec{E} \perp d\vec{a}$). The flux is the same through each cap in <u>magnitude</u>, but opposite in <u>sign</u> because the outward sense of $d\vec{a}$ is reversed in one relative to the other.

Gauss's law is nicely expressed graphically by <u>field-line</u> diagrams, where curves ("lines") are drawn that are everywhere tangent to the electric field. By making sure that the curves only terminate on charges ("as "sources" and "sinks") we have a representation of flux as the <u>number</u> of lines that cross a surface with a particular sense (into vs. out-of). Here's the field-line picture of a monopole with two surfaces crossed "positively" by the same number of lines



Contrast this with the field line diagram of a dipole, where the net line crossings (# positive − # negative) is zero when both charges are inside the surface:



● = positive crossing

○ = negative crossing

⊕ = source 10 lines

⊖ = sink 10 lines