# Lecture 41

The realization that electric and magnetic fields in empty space could by themselves be the basis of dynamical phenomena was one of the most important achievements of 19th century physics. As first analyzed by Maxwell, using the equations he distilled from the experiments of Faraday and predecessors, these phenomena always take the form of waves. The coincidence between their propagation velocity and that of light could not be overlooked, so in addition there emerged a physical model of light. And once the model of light was in hand, it was not long before the phenomenon was harnessed in completely novel ways (radio, etc.).

In this last lecture we will see how electromagnetic wave oscillations emerge very directly from Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{E}+\dot{\vec{B}}=0 \qquad \vec{\nabla}\times\vec{B}-\frac{1}{c^2}\dot{\vec{E}}=0$$

We will consider a special case, where both $\vec{E}$ and $\vec{B}$ vary in only one direction of space, which we take to be the z-coordinate. Both fields will also vary with time. Another simplification will be to assume that only the x-component of $\vec{E}$ is nonzero. Our starting assumption is therefore

$$\vec{E} = E_x(z,t)\,\hat{x}$$

This clearly satisfies $\vec{\nabla}\cdot\vec{E} = 0$, since there is only an x-component and it does <u>not</u> vary with x.

Next we consider the curl of $\vec{E}$. This has only a y-component:

$$\vec{\nabla} \times \vec{E} = \left(\frac{\partial E_x}{\partial z}\right) \hat{y}$$

$$= (\text{function of } z \,\&\, t)\, \hat{y}$$

$$= -\dot{\vec{B}} \quad (\text{by Maxwell})$$

From this we see that $\vec{B}$ has a y-component which is a function of z and t. We will assume the other components are zero:

$$\vec{B} = B_y(z,t)\, \hat{y}$$

With these forms for $\vec{E}$ and $\vec{B}$ and the equation

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \qquad (1)$$

we have satisfied two of

Maxwell's equations. But $\vec{\nabla}\cdot\vec{B}=0$ 5
is also satisfied, since $\vec{B}$ only
has a y-component and yet
does not vary in y.

That leaves just one more
Maxwell equation (Ampere):

$$\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\dot{\vec{E}}$$

Only one component of this vector
equation could be nonzero: the
x-component:

$$(\vec{\nabla}\times\vec{B})_x = \left(-\frac{\partial B_y}{\partial z}\right) = \frac{1}{c^2}\frac{\partial E_x}{\partial t} \quad (2)$$

We will have satisfied all
of Maxwell's equations if we

can find two functions of two variables

$$E_x(z,t)$$

$$B_y(z,t)$$

that satisfy the two equations 1 and 2. We can derive one equation just involving $E_x$ like this:

$$\frac{\partial}{\partial z}(1): \quad \frac{\partial^2 E_x}{\partial z^2} = -\frac{\partial^2 B_y}{\partial z \partial t}$$

$$= -\frac{\partial^2 B_y}{\partial t \partial z}$$

$$\frac{\partial}{\partial t}(2): \quad = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}$$

$$\Rightarrow \quad \frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}$$

This is the "wave equation" in one spatial dimension ($z$), which shows up in many other dynamical systems (vibrating strings, ...). The most general solution is

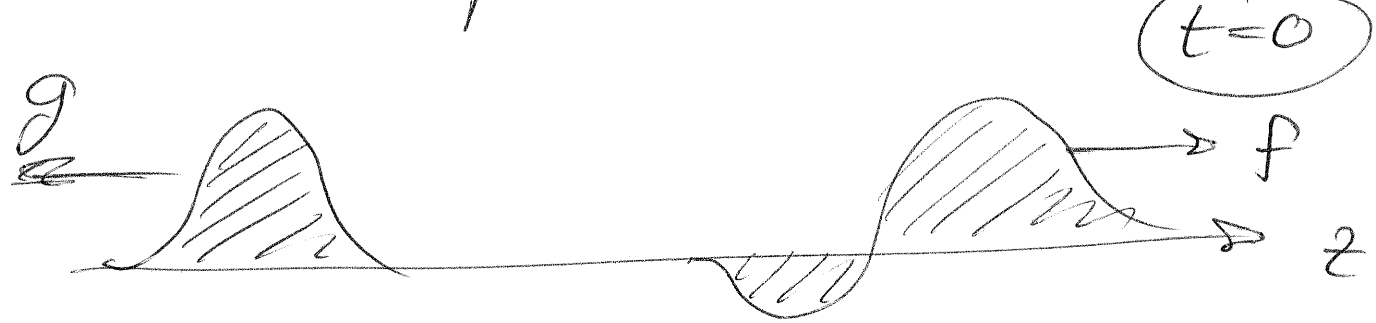
$$E_x(z,t) = f(z-ct) + g(z+ct)$$

$$\frac{\partial E_x}{\partial z} = f'(z-ct) + g'(z+ct)$$

$$\frac{\partial^2 E_x}{\partial z^2} = f''(z-ct) + g''(z+ct)$$

$$\frac{\partial^2 E_x}{\partial t^2} = c^2 f''(z-ct) + c^2 g''(z+ct)$$

The functions $f$ and $g$ are completely arbitrary and represent wave forms moving in the $+z$ and $-z$ directions, respectively, both with speed $c$:




An important special case is a periodic wave with wavelength $\lambda$:

$$f(z) = A\sin(2\pi z/\lambda)$$

$$E_x = f(z-ct) = A\sin\left(2\pi \frac{z}{\lambda} - 2\pi \nu t\right)$$

$$\nu = \frac{c}{\lambda}$$

$$= A\sin(kz - \omega t)$$

$$\left( k = \frac{2\pi}{\lambda}, \quad \omega = 2\pi\nu \right)$$

Since

$$\frac{\partial E_x}{\partial z} = k A \cos(kz - \omega t)$$

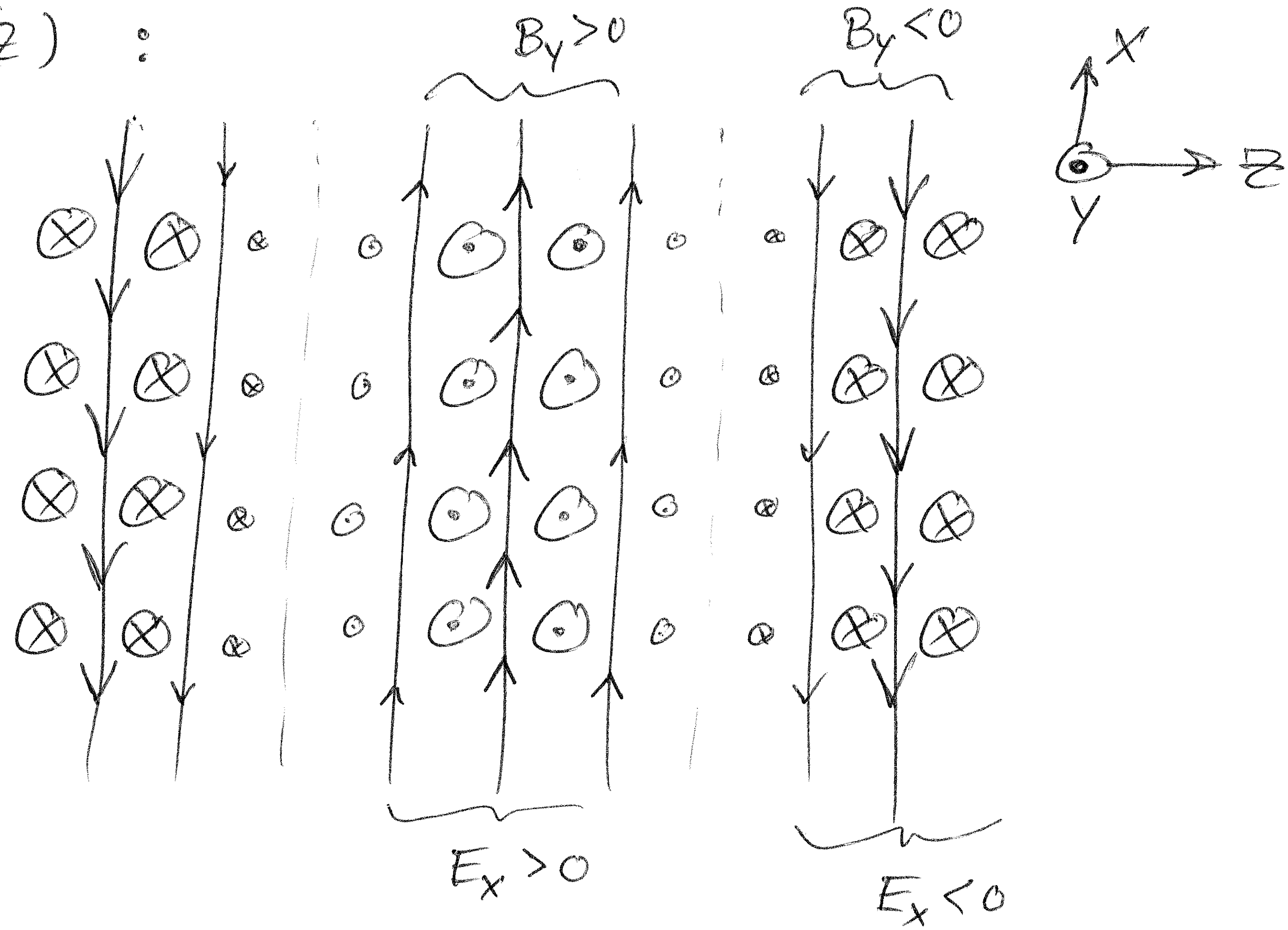
$$= -\frac{\partial}{\partial t}\left( \frac{k}{\omega} A \sin(kz - \omega t)\right)$$

from equation (1) we see

$$B_y(z,t) = \frac{k}{\omega} A \sin(kz - \omega t)$$

$$= \frac{1}{c} A \sin(kz - \omega t)$$

So the magnetic field has exactly the same form as the electric field. The following drawing reminds us that the two fields are perpendicular to each other and also perpendicular to the direction of propagation ($\hat{z}$) :




This picture is also consistent with the direction of energy flow as given by the Poynting vector:

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{1}{\mu_0} (E_x B_y) \hat{z}$$

$$= \frac{1}{\mu_0 c} A^2 \sin^2(kz - \omega t) \hat{z}$$