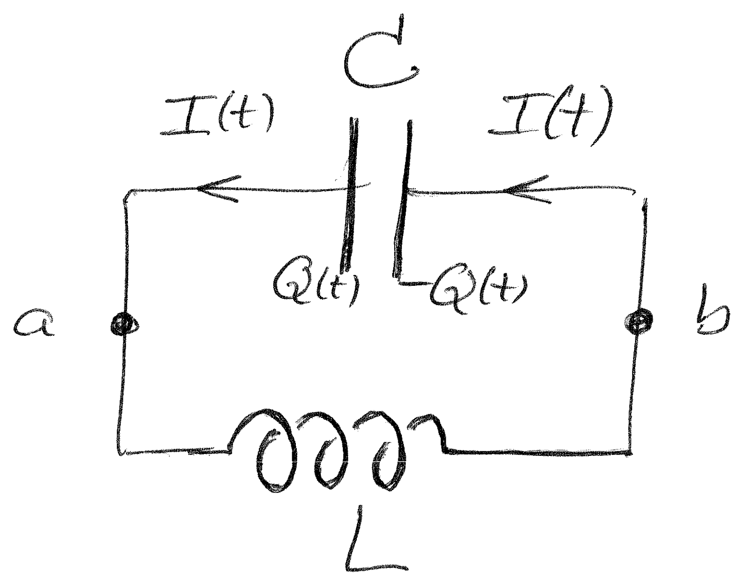
Lecture 40

A circuit that combines the two forms of energy storage — electric and magnetic — is a capacitor connected to an inductor:



Initially we'll have the capacitor charged, $Q(0) = Q_0$, and current first starts flowing at $t=0$ when we close a switch, so $I(0) = 0$.

Applying the loop rule to our circuit:

$$V_a - V_b = L\frac{dI}{dt} = Q/C$$

And, since $I = -\frac{dQ}{dt}$,

$$L\left(-\frac{d^2Q}{dt^2}\right) = Q/C$$

or $$\frac{d^2Q}{dt^2} = -\frac{Q}{LC}$$

The combination LC is, dimensionally, the square of a time:

$$LC = \tau^2$$

We also know that L and C

are μ_0 and ϵ_0 times lengths associated with the device geometry:

$$L = \mu_0 \lambda_L \quad C = \epsilon_0 \lambda_C .$$

$$\Rightarrow (\mu_0 \lambda_L)(\epsilon_0 \lambda_C) = \tau^2$$

$$\frac{\sqrt{\lambda_L \lambda_C}}{\tau} = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$$

So the geometric mean of the lengths, divided by the time τ, is the speed of light.

We can also let $1/LC$ define the square of a frequency, or angular frequency:

$$\frac{1}{LC} = \omega^2$$

Our equation for $Q(t)$ is now

$$\ddot{Q} = -\omega^2 Q$$

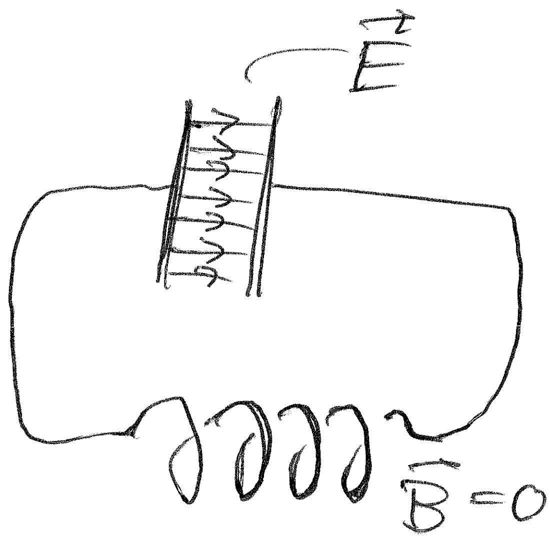
with general solution

$$Q(t) = A\cos(\omega t - \phi)$$

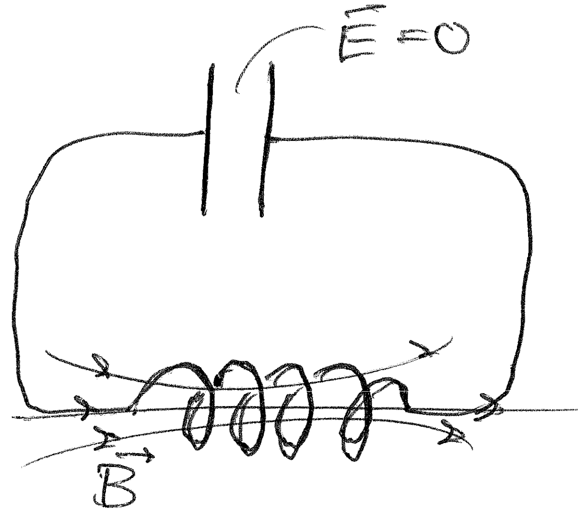
where A and ϕ are arbitrary constants. Since we want $\dot{Q}(0) = -I(0) = 0$, $\phi = 0$, and therefore $Q(0) = A = Q_0$. The solution with our initial conditions is therefore

$$Q(t) = Q_0 \cos(\omega t) .$$

At times $t = 0, \pi/\omega, 2\pi/\omega,$ etc. the charge amplitude is at a maximum or minimum and $I = 0$; the energy is then purely electric and stored in the capacitor:



Between these times $Q = 0$ and I has a maximum or minimum amplitude (clockwise vs. counterclockwise); the energy is then purely magnetic and stored in the inductor:



In rough terms, the time between these extremes is τ, and the distance between the devices is of order λ_c or λ_L or something of that order. The speed of propagation is therefore of order

$$\sqrt{\lambda_c \lambda_L}/\tau = c.$$

We will finish up the course ~~by~~ with Maxwell's discovery that these oscillations can happen in <u>empty space</u>, with energy being transformed between ~~the~~ its electric and magnetic forms in an intricate pattern. As this is happening, energy is also <u>flowing</u> through space. We will finish this lecture by deriving energy flow vector field.

Consider a volume V in space with closed surface S. We will compute the time-rate-of-change of electromagnetic energy in-side V and relate it to a quantity that flows through S.

$$U_V = \int_V \Big(\underbrace{\frac{\epsilon_0}{2} \vec{E}\cdot\vec{E}}_{\text{electric energy density}} + \underbrace{\frac{1}{2\mu_0} \vec{B}\cdot\vec{B}}_{\text{magnetic energy density}} \Big) d^3r$$

$$\frac{dU_V}{dt} = \int_V \Big(\epsilon_0 \vec{E}\cdot\dot{\vec{E}} + \frac{1}{\mu_0} \vec{B}\cdot\dot{\vec{B}} \Big) d^3r$$

Substitute $\dot{\vec{E}}$ and $\dot{\vec{B}}$ from Maxwell's equations:

$$\frac{dU_V}{dt} = \int_V \Big(\epsilon_0 \vec{E}\cdot c^2 \vec{\nabla}\times\vec{B} + \frac{1}{\mu_0} \vec{B}\cdot\big(-\vec{\nabla}\times\vec{E}\big)\Big) d^3r$$

$$= \frac{1}{\mu_0} \int_V \underbrace{\left(\vec{E} \cdot \vec{\nabla} \times \vec{B} - \vec{B} \cdot \vec{\nabla} \times \vec{E} \right)}_{-\vec{\nabla} \cdot (\vec{E} \times \vec{B})} d^3r$$

This is the volume integral of a divergence, to which we can apply the divergence theorem.

Define

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \quad ,$$

then

$$\frac{dU_V}{dt} = - \oint_S \vec{S} \cdot d\vec{a} \quad .$$

Thus $\vec{S}$ represents the flux of electromagnetic energy. Its units are energy per unit area and time, also called "intensity". The direction of $\vec{S}$ is the direction of energy flow: when $\vec{S}$ is parallel to the outward normals $d\vec{a}$ then the surface integral is positive and energy leaves the volume. The vector field $\vec{S}$ is named after John Poynting ($\vec{S}$ "poynts" in the direction of energy flow).