Lecture 3

A very direct way to see the "reality" of the electric field is through the electric energy density defined by it:

$$u(\vec{r}) = K' \vec{E}(\vec{r}) \cdot \vec{E}(\vec{r})$$
$$= K' |E(\vec{r})|^2$$

Why this is an energy density, or even relevant to the topic of electricity, is not at all obvious at this point. But what we do know, after doing HW 1, is that the integral of $u(\vec{r})$ over all of space gives an energy that exactly matches the sum of pair-wise Coulomb energies we derived in lecture 2. This is interesting because it suggests that, physically, it is the electric field that stores the energy in a system of charges.

The energy of two +q charges at a certain distance is higher than that of a +q/−q pair at the same distance:

$$K\frac{q^2}{R} \quad \text{vs.} \quad K\frac{q(-q)}{R}$$

To see what this means in terms of the electric field (energy storage), consider how the field behaves far from both charges. When both charges are +q the ~~two~~ electric fields created by the point charges approximately add. The net electric field, far away, is like the field of a single $+2q$ charge. By contrast, when the charges have opposite sign the electric fields approximately cancel, resulting in a net field that is strongly diminished. Far away, at least, we see that less energy is stored in the density $u(\vec{r})$ when the charges have opposite sign.

We will work out the near cancellation of fields for unlike charges in quantitative detail. The superposition of two "monopole" fields, created by point charges of opposite sign, is called a "dipole" field.

point charge electric field:

$$\vec{E}(\vec{r};\ \underbrace{q,\vec{s}}_{\text{source}}) = \frac{Kq}{|\vec{r}-\vec{s}|^3}(\vec{r}-\vec{s})$$

field point

charge at origin ($\vec{s}=0$):

$$\vec{E}(\vec{r}) = Kq\,\frac{\vec{r}}{|\vec{r}|^3} = Kq\,\frac{\hat{r}}{|\vec{r}|^2}$$

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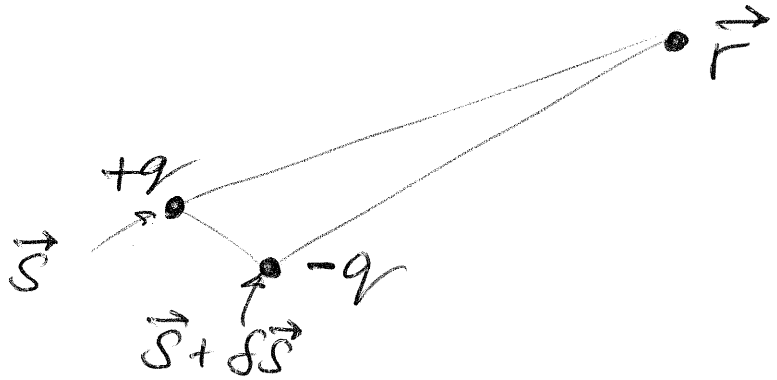
dipole field:

$$\vec{E}(\vec{r};\ q,\vec{s}) + \vec{E}(\vec{r};\ -q,\ \vec{s}+\delta\vec{s}) = \vec{D}$$

$\delta\vec{s}$ = separation of charges



We assume $|\delta\vec{s}|$ is <u>small</u> compared to other distances ($|\vec{r}-\vec{s}|$, $|\vec{r}-\vec{s}-\delta\vec{s}|$) so that we can use calculus to get approximate expressions.

$$\vec{D} = Kq\frac{\vec{r}-\vec{s}}{|\vec{r}-\vec{s}|^3} - Kq\frac{\vec{r}-\vec{s}-\delta\vec{s}}{|\vec{r}-\vec{s}-\delta\vec{s}|^3}$$

Vector calculus approximation of

$$\frac{1}{|\vec{a}+\vec{b}|^3}, \quad \text{for } |\vec{b}| \to 0$$

More generally, if f is a scalar-valued function of a vector variable, then

$$f(\vec{a}+\vec{b}) = f(\vec{a}) + \vec{b}\cdot\vec{\nabla}f(\vec{a}) + \cdots$$

This is the vector notation for

$$f(a_x+b_x\,,\,a_y+b_y\,,\,a_z+b_z) =$$

$$f(a_x, a_y, a_z) + b_x \frac{\partial f}{\partial a_x} + b_y \frac{\partial f}{\partial a_y} + b_z \frac{\partial f}{\partial a_z}$$

$$+ \cdots$$

$$\vec{\nabla} f = \frac{\partial f}{\partial a_x}\hat{x} + \frac{\partial f}{\partial a_y}\hat{y} + \frac{\partial f}{\partial a_z}\hat{z}$$

= "gradient of f"

Combine this Taylor series formula with the chain rule:

g = scalar-valued function of scalar variable

$$g(f(\vec{a}+\vec{b})) = g(f(\vec{a})) + \frac{dg}{df}(\vec{b}\cdot\vec{\nabla} f)$$

$$+ \cdots$$

Here is a very basic gradient calculation:

$$f(\vec{a}) = |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

$$\frac{\partial f}{\partial a_x} = \frac{1}{2} \frac{2a_x}{\sqrt{a_x^2 + a_y^2 + a_z^2}} = \frac{a_x}{|\vec{a}|}$$

$$\vec{\nabla}|\vec{a}| = \vec{\nabla} f = \frac{\vec{a}}{|\vec{a}|} = \text{unit vector}$$

Using this form for f, and

$g(f) = \frac{1}{f^3}$, we get:

$$\frac{1}{|\vec{a}+\vec{b}|^3} = \frac{1}{|\vec{a}|^3} + \left(\frac{-3}{f^4}\right)\left(\frac{\vec{a}}{|\vec{a}|} \cdot \vec{b}\right) + \cdots$$

$$= \frac{1}{|\vec{a}|^3} - 3\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^5} + \cdots$$

Back to the dipole field:

$$\frac{1}{|\underbrace{\vec{r}-\vec{s}}_{\vec{a}}\underbrace{-\delta\vec{s}}_{\vec{b}}|^3} = \frac{1}{|\vec{r}-\vec{s}|^3} + 3\frac{(\vec{r}-\vec{s})\cdot\delta\vec{s}}{|\vec{r}-\vec{s}|^5}$$

$$+ \cdots$$

Combine with rest of expression for $\vec{D}$, omitting terms second order in $\delta\vec{s}$:

cancel

$$\vec{D} = Kq\frac{\vec{r}-\vec{s}}{|\vec{r}-\vec{s}|^3} - Kq\frac{\vec{r}-\vec{s}+\delta\vec{s}}{|\vec{r}-\vec{s}|^3}$$

$$-3Kq\frac{\vec{r}-\vec{s}}{|\vec{r}-\vec{s}|^5}(\vec{r}-\vec{s})\cdot\delta\vec{s}$$

$$+ \cdots$$

$$= \frac{Kq}{|\vec{r}-\vec{s}|^3}\left(\delta\vec{s} - 3\frac{\delta\vec{s}\cdot(\vec{r}-\vec{s})(\vec{r}-\vec{s})}{|\vec{r}-\vec{s}|^2}\right)$$

Place +q charge at $\vec{s}=0$:

$$\vec{D} = \frac{Kq}{r^3}\left(\delta\vec{s} - 3(\delta\vec{s}\cdot\hat{r})\,\hat{r}\right)$$

$$\left(|\vec{r}-\vec{s}| \longrightarrow r \quad , \quad \frac{\vec{r}-\vec{s}}{|\vec{r}-\vec{s}|} \longrightarrow \hat{r} \right)$$