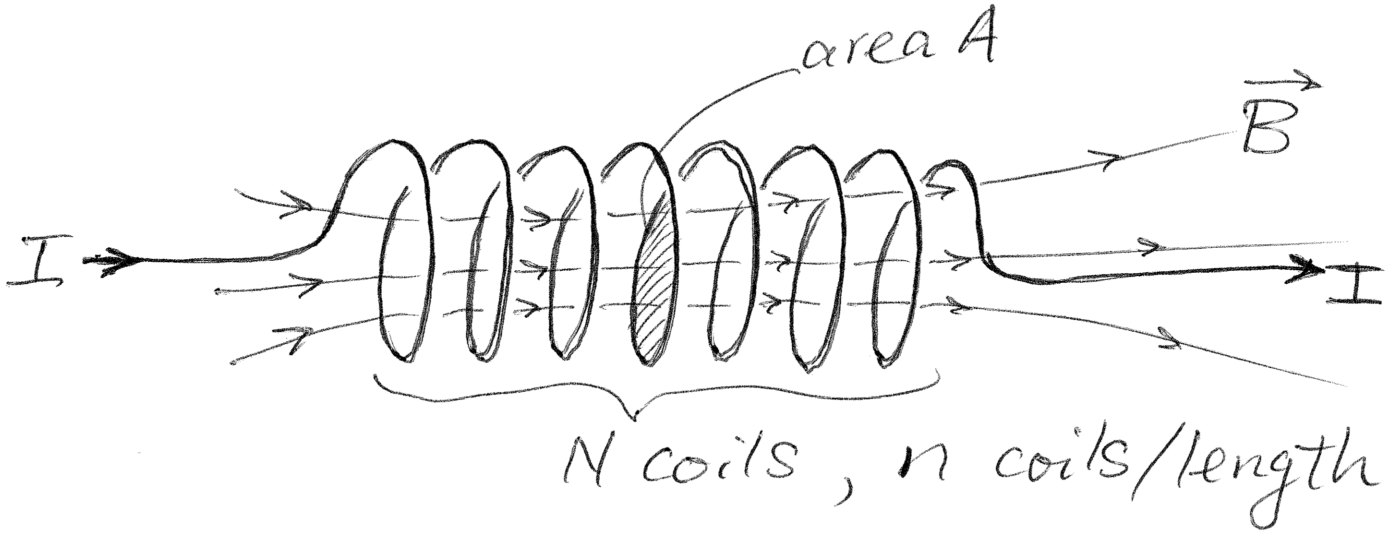
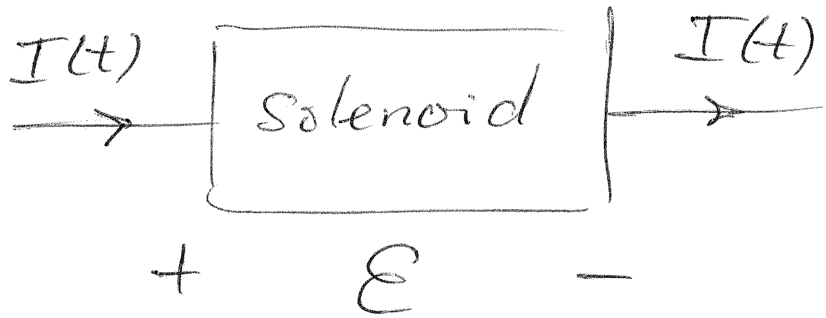
Lecture 39

Today we'll see how induced EMFs play a role in the design of circuits. Devices called "inductors" amplify and isolate induced electric fields. The canonical inductor model is the solenoid:



The magnetic field generated by the current is very uniform within the solenoid and decays rapidly with distance on the outside (like a dipole), thereby confining inductance effects to the solenoid itself.

By Ampere's law, the magnetic field magnitude on the inside is $B = \mu_0 n I$ (lecture 30). Now suppose I is <u>increasing</u> with time, so the flux of $\vec{B}$ through each coil is also increasing. By Lenz's law, there will be an EMF around each turn driving an induced current whose magnetic flux tries to keep the net flux constant. The direction of the induced current is therefore such that it opposes the <u>change</u> in the applied current, and overall the solenoid appears to the rest of the circuit as an EMF with the following polarity:



$$\frac{dI}{dt} > 0$$

The net EMF $\mathcal{E}$ is the sum of EMFs around all N coils, or equivalently, that associated with changing flux through the combined area $N \cdot A$:

$$\mathcal{E} = \frac{d}{dt}\underbrace{(NAB)}_{\Phi_B} = \frac{d}{dt}(\underbrace{NA\mu_0 n}_{\text{const.}} I)$$

$$= L\frac{dI}{dt}$$

$$L = \Phi_B / I = NA\mu_0 n \quad \text{(solenoid)}$$

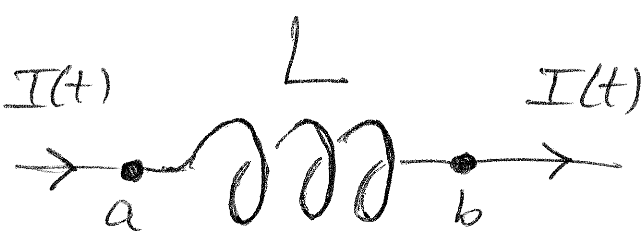
The constant of proportionality L between flux and current is called the "inductance". Inductance has units

$$[L] = \frac{[\mathcal{E}]}{[\dot{I}]} = \frac{V}{A/s} = \Omega \cdot s$$

This combination is called "Henry" (H) after the American physicist Joseph Henry. From the solenoid inductance formula we see that μ_0 has units of H/m:

$$\mu_0 \cong 1.26\ \mu H/m$$

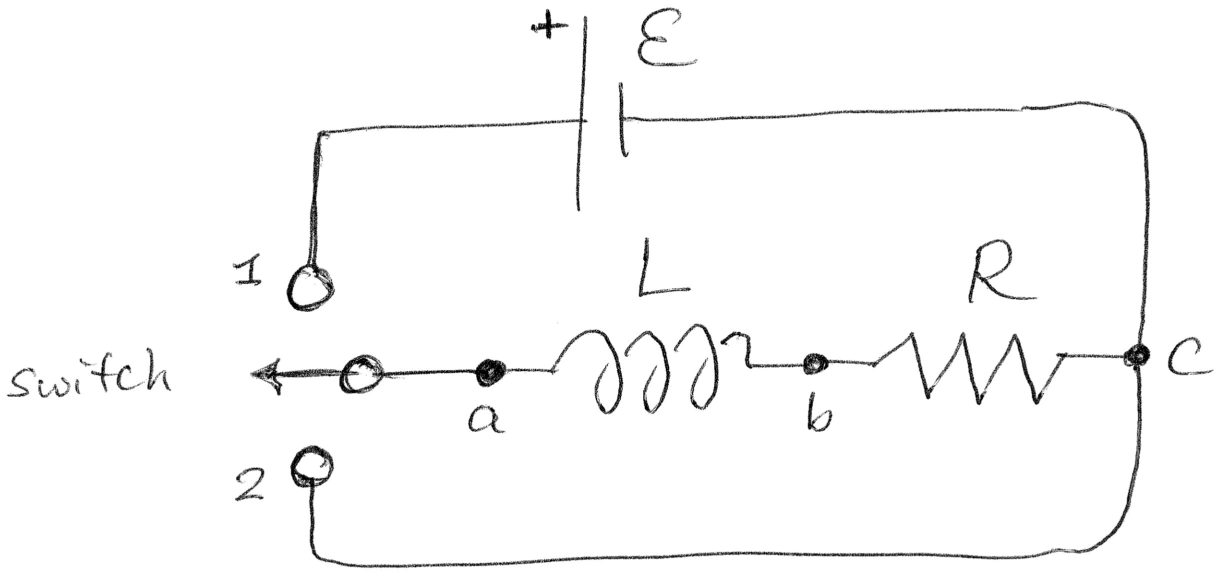
In a circuit, an ~~d~~ inductor (such as a solenoid) is represented as a device that creates a potential difference between its ends depending on the time rate of change in the applied current:



$$V_a - V_b = L\frac{dI}{dt}$$

This formula is valid also when $\frac{dI}{dt}$ is negative. In that case, the flux is decreasing so the induced EMFs are in the opposite direction (so terminal "b" looks like the positive terminal of a battery).

Here's a simple circuit that includes an inductor:



We start with the switch in the (1) position. Previously, ignoring the effects of inductance (inductor = idealized wire) we would have concluded the current jumps instantaneously to the value $\mathcal{E}/R$ in the upper loop when the switch is closed. To see what actually happens we apply Kirchhoff's loop rule with inductor's EMF included:

$$V_a - V_b = L\frac{dI}{dt}$$

$$V_b - V_c = IR$$

$$V_c - V_a = -\mathcal{E}$$

$$\Rightarrow \quad 0 = L\dot{I} + RI - \mathcal{E}$$

$$\dot{I} + (R/L)I = \mathcal{E}/L$$

With the inductor in place, at the instant the switch is thrown and $I=0$ we see that the current increases at the <u>finite</u> rate $\dot{I}(0) = \mathcal{E}/L$. Solving the differential equation is virtually the same as the case of the RC circuit (lecture 19) with $I(t)$

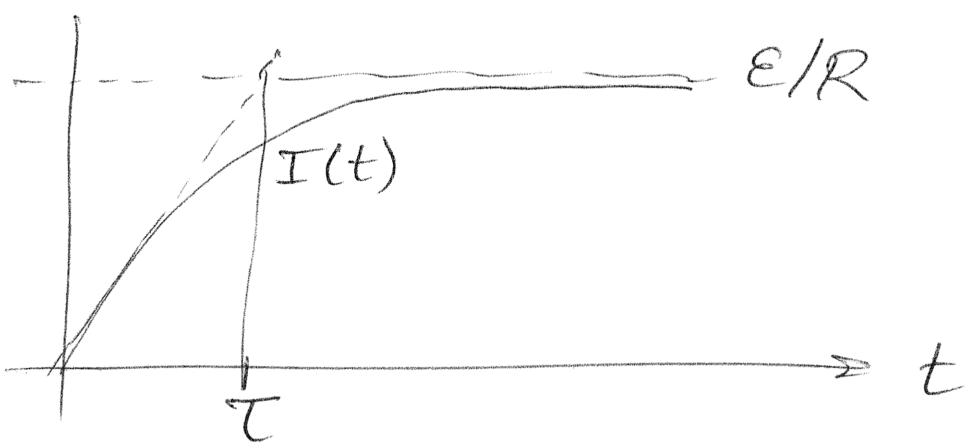
replacing $Q(t)$:

$$I(t) = \mathcal{E}/R \left(1 - e^{-(R/L)t}\right)$$

$$\sim \mathcal{E}/R\left(1-\left(1-R/L\,t\right)\right) = \frac{\mathcal{E}}{L}\,t$$

<u>small t</u>

$$\sim \mathcal{E}/R$$ <u>large t</u>



$$\tau = L/R = \text{"time constant"}$$

The constant τ gives the finite time scale for the current to "turn on".

After a steady current $\mathcal{E}/R$ has been running for awhile we throw the switch into position 2. This doesn't happen instantaneously, and it would appear that there is a brief time when no current is flowing at all! But this corresponds to an enormous $\frac{dI}{dt}$ and equally enormous inductive EMF $L\dot{I}$. In fact, the potential between the poles of the switch will be so enormous that air is ionized, thereby providing a mechanism for current flow and keeping $\dot{I}$ finite. This explains the sparking we see when we try to interrupt the current in an inductor. Once the switch is in position (2) the equation for the current is

$$\dot{I} + (R/L)\, I = 0$$

with solution

$$I(t) = I(0)\, e^{-t/\tau}.$$

As the current flows in the lower loop (the battery is out of the picture) energy is being consumed-by/dissipated-in the resistor. Where is this energy coming from? When we encountered this situation with the capacitor, we argued the energy was originally stored in the electric field of the capacitor. In the case of the inductor, energy is stored in the magnetic field. All of the energy concepts associated with electric fields have magnetic counterparts; we will simply list them here:

electric	magnetic
$dU = V\,dQ$ $= \left(\frac{Q}{C}\right) dQ$	$dU = V\,dQ$ $= (L\dot{I})(I\,dt)$ $= (LI)(dI)$
$U = \frac{1}{2}\frac{Q^2}{C}$	$U = \frac{1}{2} L I^2$
$u = \frac{\epsilon_0}{2} \lvert\vec{E}\rvert^2$	$u = \frac{1}{2\mu_0} \lvert\vec{B}\rvert^2$