Lecture 37

today we complete our "bottom-up" (Coulomb's Law + general principles) construction of the Maxwell equations by restoring the sources:

(1) $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ (2) $\vec{\nabla}\cdot\vec{B} = 0$

(3) $\vec{\nabla}\times\vec{E} + \dfrac{\partial\vec{B}}{\partial t} = 0$

(4) $\vec{\nabla}\times\vec{B} - \dfrac{1}{c^2}\dfrac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$

The sources are valid for the static equations — will they continue to be valid in the dynamic case, when $\vec{E}$ and $\vec{B}$ vary in time? By going into a moving frame, which makes static fields look dynamic, we will see that for this to be true ρ and $\vec{j}$ have to transform in a certain way. For frames related by the same boost we have been using in the last lectures, invariance of the Maxwell equations requires the following transformation of charge and current density:

$$\rho' = \gamma(\rho - \tfrac{v}{c^2} j_x)$$

$$j_x' = \gamma(j_x - v\rho)$$

$$j_y' = j_y$$

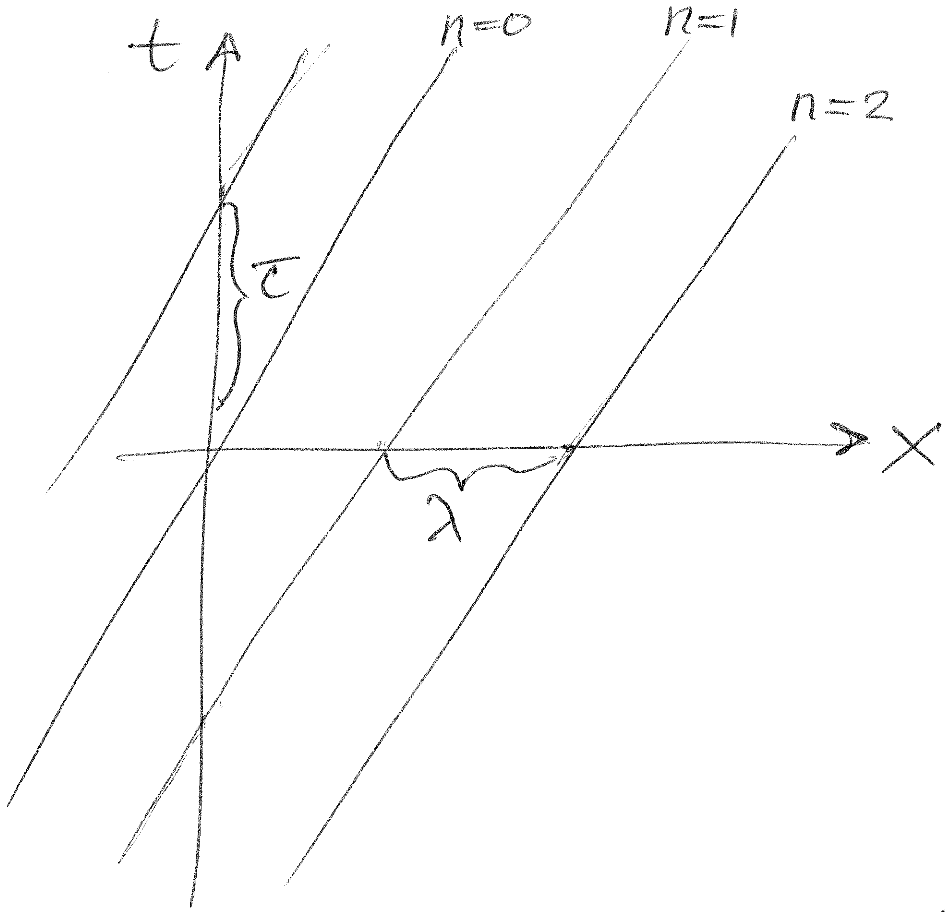
$$j_z' = j_z$$

Verifying this is an easy extension of HW 11 which did not have sources. For example, when transforming

$$\vec{\nabla}' \times \vec{B}' - \frac{1}{c^2}\frac{\partial \vec{E}'}{\partial t'} = \mu_0 \vec{j}'$$

you would (1) replace partial derivatives by their transformed counterparts, (2) replace primed with unprimed fields, and (3) replace $\vec{j}'$ with ρ and $\vec{j}$ as necessary (different components are transformed differently).

From the transformation we see that ρ and $\vec{j}$ together make a 4-vector. To see this more directly we will analyze a simple model of a source, where identical charges q move with equal velocity and are evenly arranged in space. The sketch on the next page shows the world lines of the particles:



Not shown is the spacing of the particles in y and z, but say there is one particle per area A in those dimensions; then

$$\rho = \frac{q}{\lambda A} \qquad j_x = \frac{q}{\tau A} .$$

The world lines are described by

the equations

$$x/\lambda - t/\tau = n$$

where n is an integer that labels the world line. In a moving frame each ~~[illegible]~~ line keeps the same integer label, but the coordinates and spacing parameters will be different:

$$x/\lambda - t/\tau = x'/\lambda' - t'/\tau'$$

This is just stating the invariance of the Minkowski ~~4-vector~~ product of ~~the~~ two vectors:

$$(x, ct) \cdot (1/\lambda, 1/c\tau)$$

Minkowski

And since

$$x' = \gamma(x - vt)$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right),$$

it must be true that

$$1/\lambda' = \gamma\left(1/\lambda - \frac{v}{c^2}\,1/\tau\right)$$

$$1/\tau' = \gamma\left(1/\tau - v\,1/\lambda\right)$$

Multiplying through by q/A gives the transformation rules on page 3 for ρ and j_x.

That the components of $\vec{j}$ perpendicular to the boost direction are unchanged ($j_y' = j_y$, $j_z' = j_z$)

can be argued like this. The density of charges in space is ~~en~~ enhanced by γ due to the Lorentz contraction in one of the dimensions. On the other hand, there is also a time-dilation by γ in the rate charge is transported. These relativistic effects exactly cancel.

There is a nice <u>internal consistency</u> among the sources in Maxwell's equations and the derivatives of the fields. Take the time derivative of equation (1):

$$\frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\frac{\partial \rho}{\partial t}$$

Next take the divergence of equation (4):

$$-\frac{1}{c^2}\vec{\nabla}\cdot\frac{\partial \vec{E}}{\partial t} = \mu_0\vec{\nabla}\cdot\vec{J}$$

The order of the derivatives shouldn't matter:

$$\frac{1}{\epsilon_0}\frac{\partial \rho}{\partial t} = -c^2\mu_0\vec{\nabla}\cdot\vec{J}$$

Using the definition $\mu_0 = \frac{1}{\epsilon_0 c^2}$,

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{J},$$

the local law of <u>charge</u> <u>conservation</u>