Lecture 36

Let's summarize where we currently stand in our effort to write down laws for electric and magnetic fields that are consistent with the theory of special relativity.

First there were the laws that correctly described static situations:

$$\vec{\nabla}\cdot\vec{E}=0 \qquad \vec{\nabla}\cdot\vec{B}=0$$

$$\vec{\nabla}\times\vec{E}=0 \qquad \vec{\nabla}\times\vec{B}=0$$

We have omitted sources for now (we are in a region of space where there are none) — they will be reintroduced later.

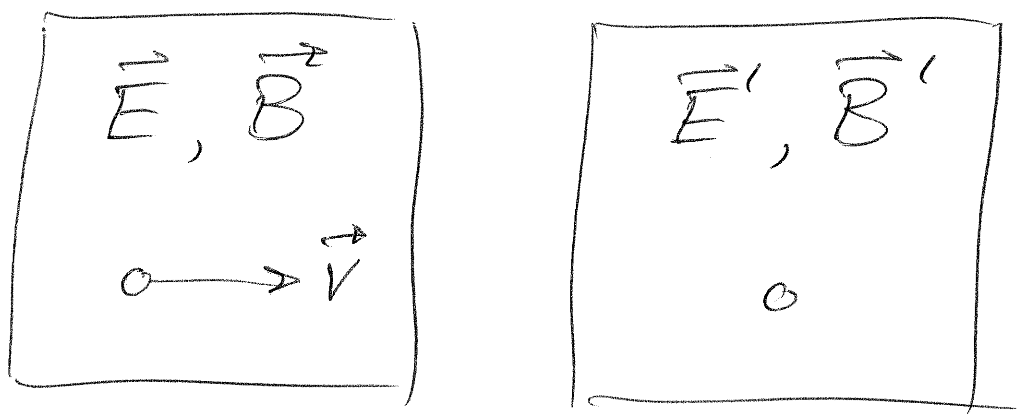
Next we considered two very <u>symmetrical</u> static situations (capacitor & solenoid fields) from a moving frame and found that by amending the static laws as follows they would continue to be valid in these special non-static situations:

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \qquad \vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = 0$$

What we need to do next is test (theoretically!) the validity of these equations for more general fields than the capacitor and solenoid examples that inspired them. The test we will apply is not exhaustive but nevertheless a very important one: are the equations Lorentz-invariant? If this works out, then we will know the equations are satisfied if in some frame the fields are static (because the equations for static fields are valid).

To apply this test we must transform both the fields and the partial derivatives. Since the form of the equations is already invariant with respect to ~~equation~~ rotations, there is no loss of generality if we choose to boost in the x-direction



$$\vec{v} = v\hat{x}$$

$$E_x' = E_x \qquad B_x' = B_x$$

$$E_y' = \gamma(E_y - vB_z) \qquad B_y' = \gamma\left(B_y + \frac{v}{c^2}E_z\right)$$

$$E_z' = \gamma(E_z + vB_y) \qquad B_z' = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

$$x = \gamma(x' + v t')$$
$$t = \gamma\left(t' + \frac{v}{c^2} x'\right)$$
$$y = y'$$
$$z = z'$$

From these we obtain the transformation of partial derivatives as in our warm-up problem:

$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t}$$
$$= \gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)$$

$$\frac{\partial}{\partial t'} = \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial t'}\frac{\partial}{\partial t}$$
$$= \gamma\left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$$

Now comes the test. Let's suppose the dynamic laws are valid in the unprimed frame — are they still valid in the primed frame? Let's start with

$$\vec{\nabla}' \times \vec{E}' + \frac{\partial \vec{B}'}{\partial t'} .$$

All three components are supposed to be zero. We'll start with the x'-component:

$$\frac{\partial E_z'}{\partial y'} - \frac{\partial E_y'}{\partial z'} + \frac{\partial B_x'}{\partial t'} =$$

$$\gamma \frac{\partial}{\partial y}(E_z + vB_y) - \gamma \frac{\partial}{\partial z}(E_y - vB_z)$$

$$+ \gamma\left(v \frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) B_x =$$

$$\gamma\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} + \frac{\partial B_x}{\partial t}\right) +$$

$$\gamma v\left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) = 0$$

since the first term is the same law (times γ) in the unprimed frame and the second term is γv times the divergence of $\vec{B}$.

The y-component works out differently:

$$\frac{\partial E_x'}{\partial z'} - \frac{\partial E_z'}{\partial x'} + \frac{\partial B_y'}{\partial t'} =$$

$$\frac{\partial E_x}{\partial z} - \gamma^2\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\left(E_z + vB_y\right)$$

$$+ \gamma^2\left(v\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right)\left(B_y + \frac{v}{c^2}E_z\right) =$$

$$\frac{\partial E_x}{\partial z} - \underbrace{\gamma^2\left(1-\frac{v^2}{c^2}\right)}_{1}\frac{\partial E_z}{\partial x} + \underbrace{\gamma^2\left(1-v^2/c^2\right)}_{1}\frac{\partial B_y}{\partial t}$$

$$= 0$$

since this is the same law in the unprimed frame.

The z-component works out in the same way as the y-component (both are perpendicular to the boost). So this shows

$$\vec{\nabla}' \times \vec{E}' + \frac{\partial \vec{B}'}{\partial t'} = 0 .$$

Next, let's test the divergence equation — a single equation:

$$\vec{\nabla}' \cdot \vec{B}' = \underbrace{\gamma\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)}_{\frac{\partial}{\partial x'}} B_x$$

$$+ \frac{\partial}{\partial y}\gamma\left(B_y + \frac{v}{c^2}E_z\right) + \frac{\partial}{\partial z}\gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

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$$\gamma\left(\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}\right)+$$

$$\gamma\frac{v}{c^2}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{\partial B_x}{\partial t}\right)=0$$

since the first term is γ times $\vec{\nabla}\cdot\vec{B}$ and the second is $\gamma\frac{v}{c^2}$ times the x-component of $\left(\vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}\right)$.

That's two of the four equations. You will check the other two in the homework!