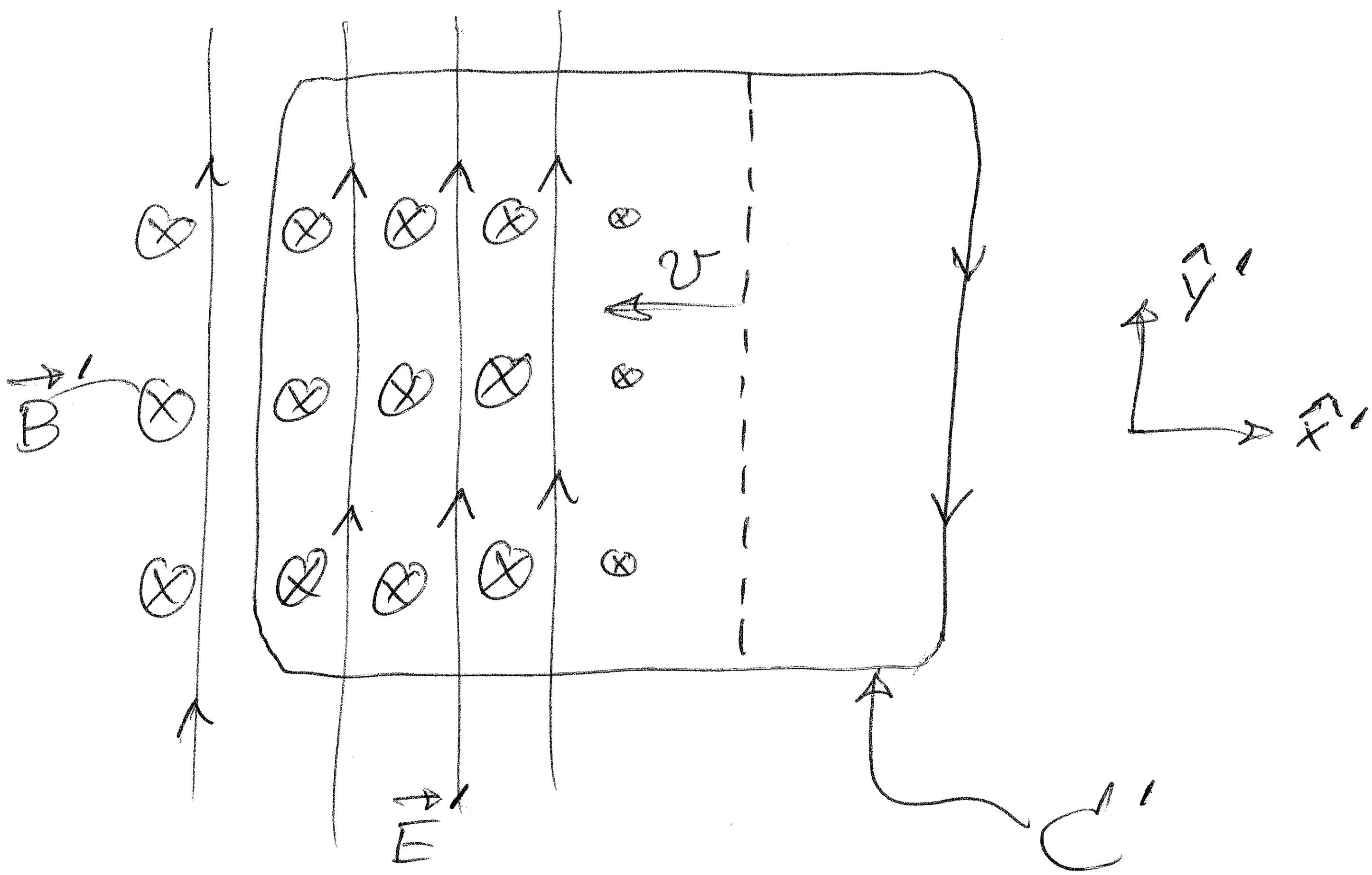
# Lecture 35

The analysis of the pure magneto-statics scenario, in a moving frame, is almost the same as the electro-static case we analyzed in the previous lecture. The roles of $\vec{E}$ and $\vec{B}$ are interchanged, there are some sign differences, and c's to keep the units correct:

$$\left.\begin{aligned} \vec{B} &= -B\hat{z} \\ \vec{E} &= 0 \\ \vec{V} &= v\hat{x} \\ \vec{V}\times\vec{B} &= vB\hat{y} \end{aligned}\right\} \qquad \begin{aligned} B_x' &= B_x = 0 \\ B_y' &= 0 \\ B_z' &= \gamma B_z = -\gamma B \\ E_x' &= E_x = 0 \\ E_y' &= \gamma v B \\ E_z' &= 0 \end{aligned}$$

$$\Longrightarrow \vec{E}' = \gamma v B\hat{y} \qquad \vec{B}' = -\gamma B\hat{z}$$

primed frame

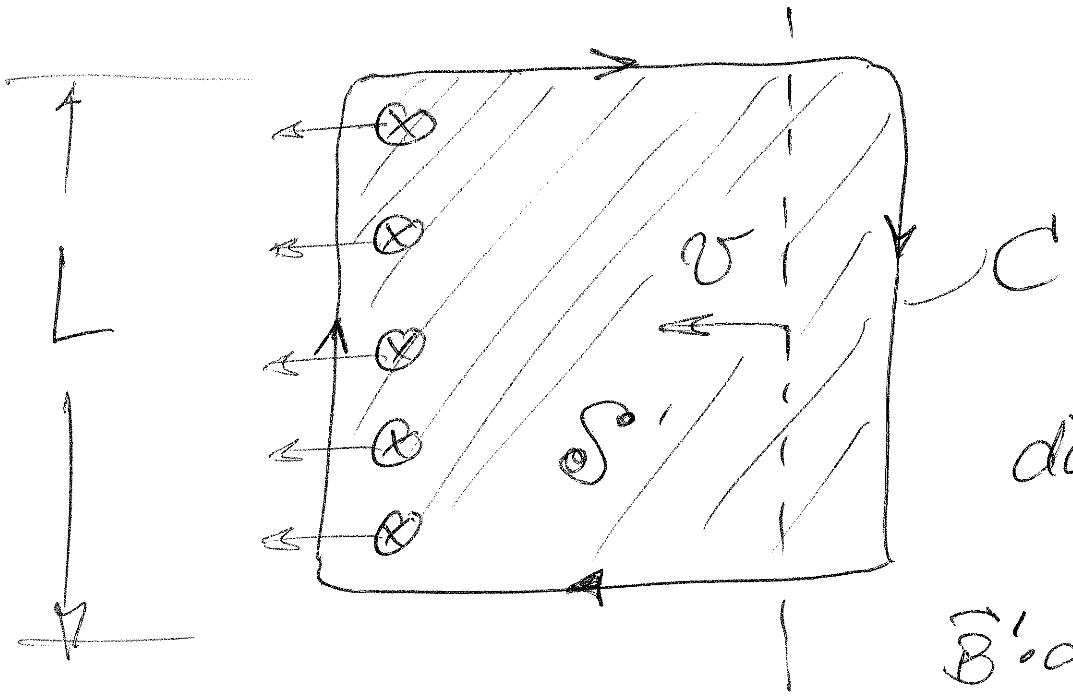



This time the circulation of $\vec{E}'$ around $C'$ is not zero (as it is in electrostatics):

$$\oint_{C'} \vec{E}'\cdot d\vec{r}' = +\gamma v B L$$

The time-dependent entity is

now the flux of <u>magnetic</u> field :




$d\vec{a}' = \otimes$

$\vec{B}' \cdot d\vec{a}' =$
$+\gamma B |d\vec{a}'|$

$$\frac{\Delta \Phi_{B'}}{\Delta t'} = \frac{-\gamma B L (v \Delta t')}{\Delta t'} = -\gamma B v L$$

This is (-1) times the discrepancy in the circulation of $\vec{E}'$, which suggest the following amendment:

$$\oint_{C'} \vec{E}' \cdot d\vec{r}\,' + \frac{d}{dt'}\left( \int_{S'} \vec{B}' \cdot d\vec{a}\,' \right) = 0$$

After bringing the time derivative inside the surface integral and using Stokes' law (exactly as earlier) we obtain:

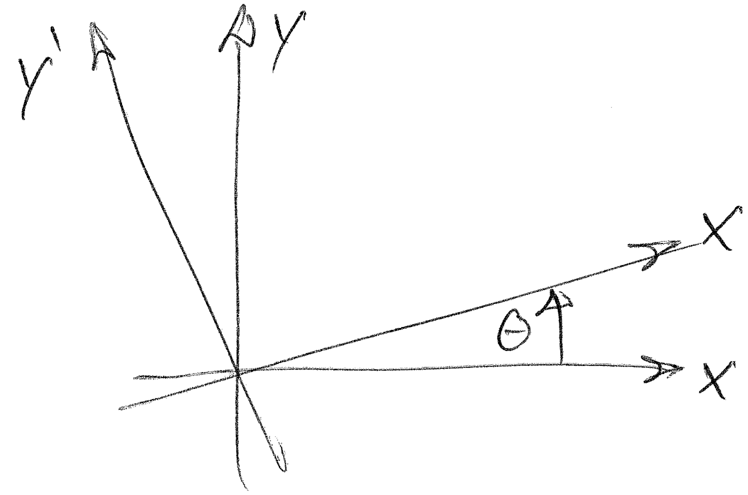
$$\int_{S'} \left( \vec{\nabla}' \times \vec{E}' + \frac{\partial \vec{B}'}{\partial t'} \right) \cdot d\vec{a}\,' = 0$$

The electrostatics law $\vec{\nabla} \times \vec{E}$ should therefore be amended as

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

We used particularly <u>simple</u> and <u>symmetric</u> field configurations to arrive at the two amendments of the laws for the circulation of $\vec{E}$ and $\vec{B}$. Before we accept them as <u>general</u> laws we should subject them to the ultimate theoretical test: invariance with respect to Lorentz transformations (i.e. the same laws apply in any inertial frame). Recently we saw how $\vec{E}$ and $\vec{B}$ transform; what remains is to work out how the space and time derivates in these laws transform.

Warm-up exercise: Suppose we have a function of two variables $x$ and $y$, say $f(x,y)$. We already know what the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ mean. But suppose we are interested in describing the $x$-$y$ plane with another pair of variables $x', y'$. As a concrete example, let $x', y'$ be coordinates in a rotated frame:




$$x = \cos\theta\, x' - \sin\theta\, y'$$
$$y = \sin\theta\, x' + \cos\theta\, y'$$

So we can think of $f$ as actually a function of $x'$ and $y'$ since $x$ and $y$ are functions of these variables:

$$\frac{\partial f}{\partial x'} = \frac{\partial f(x(x',y'),\, y(x',y'))}{\partial x'}$$

$$= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x'}$$

$$= \frac{\partial f}{\partial x} \cos\theta + \frac{\partial f}{\partial y} \sin\theta$$

Similarly:

$$\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial x}(-\sin\theta) + \frac{\partial f}{\partial y} \cos\theta$$

Since $\sin\theta$ and $\cos\theta$ are constants (the angle of rotation is fixed) we can rearrange things like this:

$$\frac{\partial}{\partial x'} f = \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right) f$$

$$\frac{\partial}{\partial y'} f = \left(-\sin\theta \frac{\partial}{\partial x} + \cos\theta \frac{\partial}{\partial y}\right) f$$

Finally, $f$ is an arbitrary function so what we've found is a transformation rule for partial derivatives.