Lecture 34

Here are the fundamental field transformation laws:

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} \qquad \vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

$$\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{V} \times \vec{B}_{\perp}) \qquad \vec{B}'_{\perp} = \gamma(\vec{B}_{\perp} - \tfrac{1}{c^2}\vec{V} \times \vec{E}_{\perp})$$

The primed frame is moving relative to the unprimed frame with velocity $\vec{V}$ and

$\parallel$ = component parallel to $\vec{V}$

$\perp$ = component perpendicular to $\vec{V}$

A simple application of these rules is to answer the question "What does a pure electrostatics scenario look like in another frame?"

"Pure electrostatics" means static charge and no current sources (e.g. point charge at rest). Suppose we have pure electrostatics in the primed frame. In that case $\vec{B}'=0$,

so:
$$\vec{B}_{\parallel} = \vec{B}'_{\parallel} = 0$$

$$\gamma\left(\vec{B}_{\perp} - \tfrac{1}{c^2}\vec{V}\times\vec{E}_{\perp}\right) = \vec{B}'_{\perp} = 0$$

These imply,

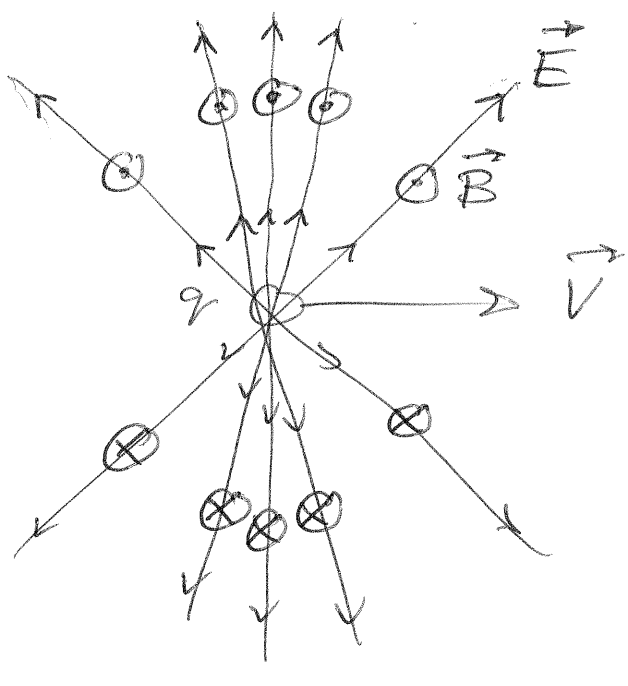
$$\vec{B}_{\perp} = \tfrac{1}{c^2}\vec{V}\times\vec{E}_{\perp} = \tfrac{1}{c^2}\vec{V}\times\vec{E}$$

or more simply yet,

$$\vec{B} = \tfrac{1}{c^2}\vec{V}\times\vec{E}$$

because the $\parallel$ component of $\vec{B}$ defined by this is zero.

We should be careful to use this only in the special case where the magnetic field is zero in some other frame — it is not true in general. Earlier in the course we worked out just the electric field $\vec{E}$ in a frame where there is a uniformly moving point charge. To obtain the magnetic field in that frame we simply calculate $\vec{B} = \frac{1}{c^2} \vec{V} \times \vec{E}$:



The magnetic field lines encircle the velocity vector of the charge ~~exa~~ according to the right-hand-rule exactly as we expect of a current source.

The same type of relationship applies between $\vec{E}$ and $\vec{B}$ when in some frame we have a pure magneto-statics problem. Setting $\vec{E}'=0$ in the transformation equations we find

$$\vec{E} = -\vec{V}\times\vec{B} \quad .$$

You would use this, for example, if ~~w~~ you wanted to know the electric field produced by a moving magnetic dipole.

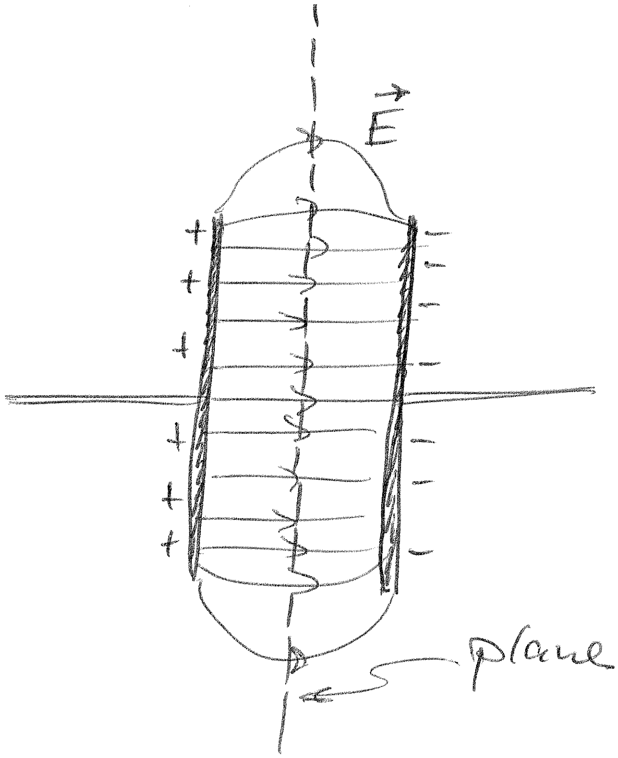
In addition to the transformation rules, the other fundamental laws for $\vec{E}$ and $\vec{B}$ we've seen so far are

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

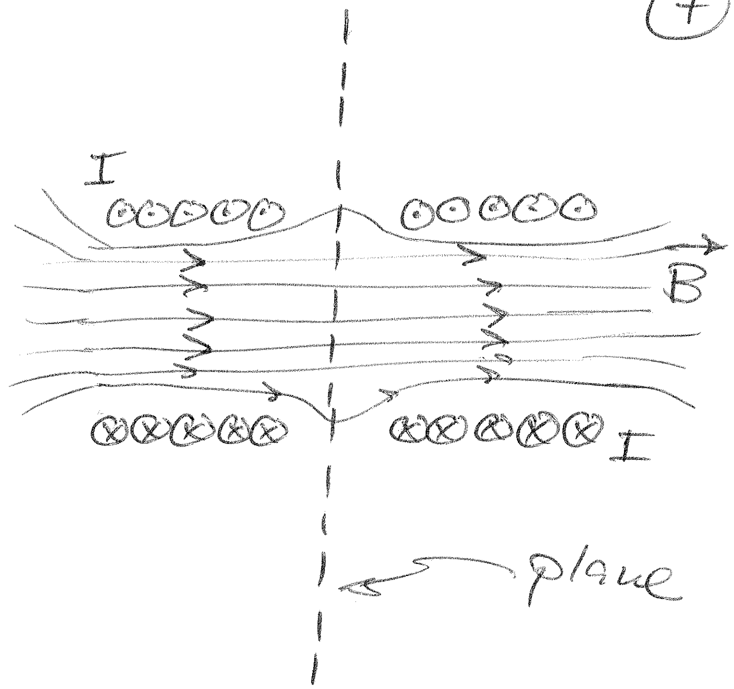
$$\vec{\nabla}\times\vec{E} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

Remember that these arose in always the context where everything $(\vec{E}, \vec{B}, \rho, \vec{J})$ is time-independent. This, and the fact that equations involving only spatial derivatives cannot be Lorentz-invariant, makes us suspect these laws need to be amended for time-dependent situations. The field transformation rules will be our guide to making these amendments.

For inspiration on the nature of the necessary amendments we will look at ~~the~~ pure magneto-static and electro-static situations from a moving frame. To keep things as simple as possible, we will look at fields that are everywhere perpendicular to a particular plane in the "static" frame (where either $\vec{E}$ or $\vec{B}$ is time-independent). The physical scenarios are sketched below:

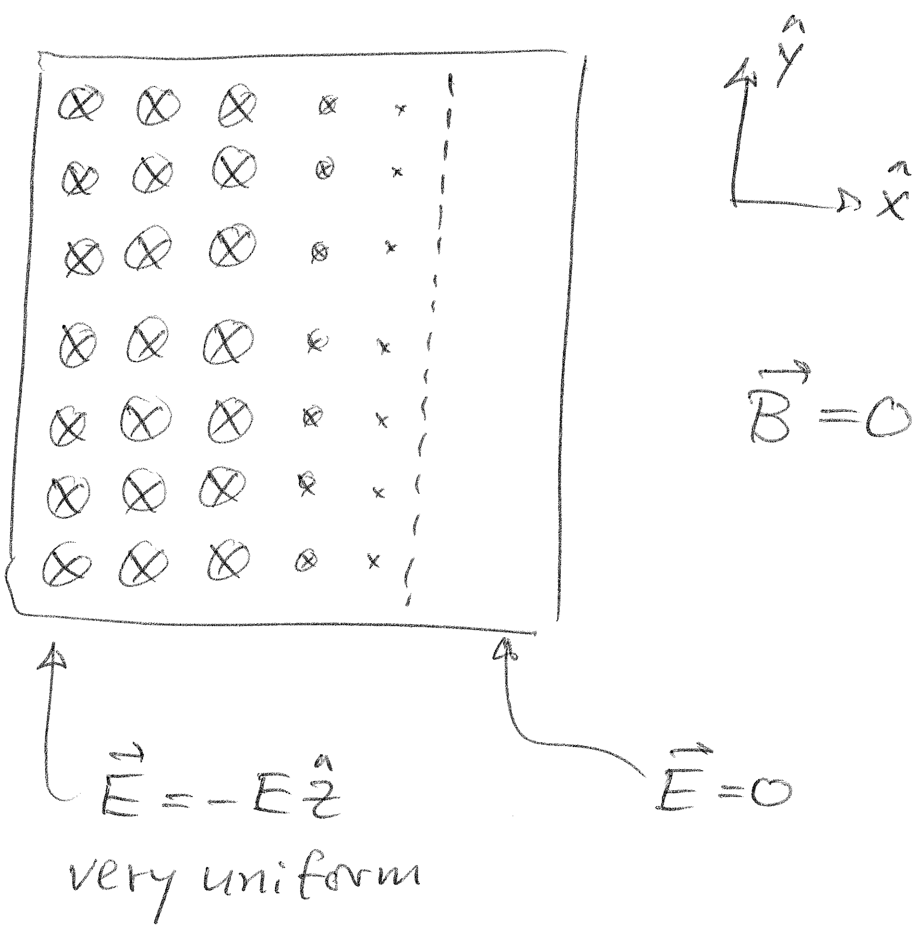


capacitor
(electrostatics)



solenoid
(magnetostatics)

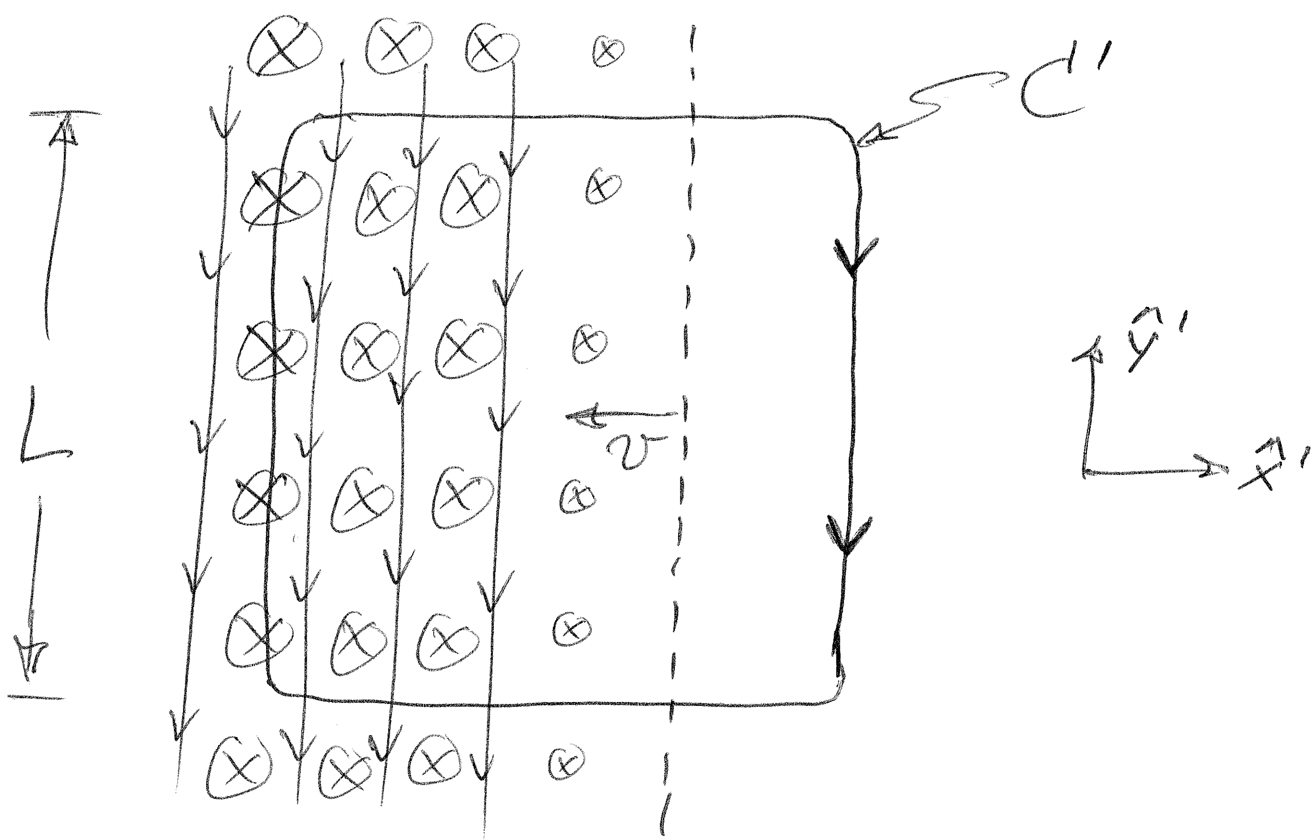
Let's start with the capacitor, where there is just an electric field in the unprimed frame. We'll look at the region of the plane where the field switches from a uniform value E deep inside the capacitor to zero far outside.



In a frame moving to the right, with velocity $\vec{V} = v\hat{x}$ relative to the capacitor rest frame, two things have changed: the field is no longer static (the "edge" of the field is moving to the left) and the field itself is transformed. Let's apply the transformation rules to the part of the field where it is very uniform and non-zero:

$$\left.\begin{aligned} \vec{E} &= -E\hat{z} \\ \vec{B} &= 0 \\ \vec{v} &= v\hat{x} \\ \vec{v}\times\vec{E} &= vE\hat{y} \end{aligned}\right\} \rightarrow \begin{aligned} E_x' &= E_x = 0 \\ E_y' &= \gamma E_y = 0 \\ E_z' &= \gamma E_z = -\gamma E \\ B_x' &= B_x = 0 \\ B_y' &= \gamma\left(B_y - \tfrac{1}{c^2}vE\right) = -\gamma \tfrac{v}{c^2}E \\ B_z' &= \gamma B_z = 0 \end{aligned}$$

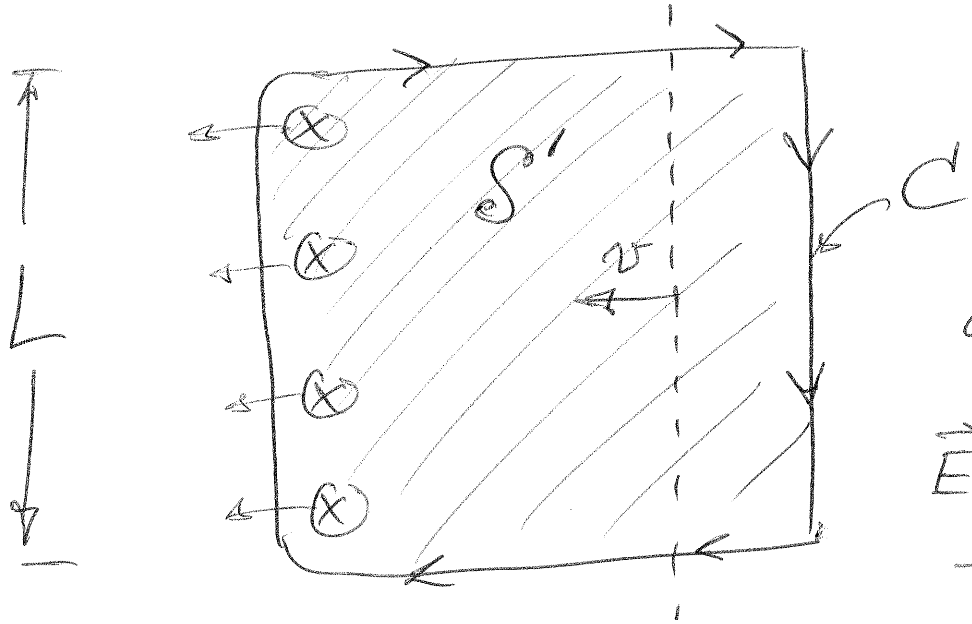
$$\vec{E}' = -\gamma E\hat{z}' \qquad \vec{B}' = -\gamma\frac{v}{c^2}E\hat{y}'$$



The diagram also shows a closed curve C' (fixed in the primed frame) with orientation shown by arrows. There are no sources of any kind in our plane, so the flux of current density through the surface S' that spans C' is zero. On the other hand, the circulation of $\vec{B}'$ around C' is not zero:

$$\oint_{C'} \vec{B}' \cdot d\vec{r}\,' = -\gamma \frac{v}{c^2} E \cdot L$$

That this failure of Ampere's law is somehow linked to time-dependence suggest we look more closely at a manifest time-dependent quantity in the primed frame: the flux of electric field through the surface S' :



$d\vec{a}'$: $\otimes$

$\vec{E}' \cdot d\vec{a}' = +\gamma E |d\vec{a}|$

$$\frac{\Delta \Phi_{E'}}{\Delta t'} = \frac{(-\gamma E) L (v \Delta t')}{\Delta t'} = -\gamma E v L$$

This is c^2 times the discrepancy we found in Ampere's law, and suggest the following amendment:

$$\oint_{C'} \vec{B}' \cdot d\vec{r}' - \frac{1}{c^2} \frac{d}{dt'} \left(\int_{S'} \vec{E}' \cdot d\vec{a}' \right) = 0$$

After the following mathematical identities,

$$\oint_{C'} \vec{B}' \cdot d\vec{r}\,' = \int_{S'} \vec{\nabla}' \times \vec{B}' \cdot d\vec{a}\,' \quad \text{(Stokes)}$$

$$\frac{d}{dt'}\left(\int_{S'} \vec{E}' \cdot d\vec{a}\,' \right) = \int_{S'} \left(\frac{\partial \vec{E}'}{\partial t'} \right) \cdot d\vec{a}\,'$$

this becomes:

$$\int_{S'} \left(\vec{\nabla}' \times \vec{B}' - \frac{1}{c^2} \frac{\partial \vec{E}'}{\partial t'} \right) \cdot d\vec{a}\,' = 0$$

A reasonable amendment of Ampere's law (in the absence of sources) is therefore

$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = 0$$