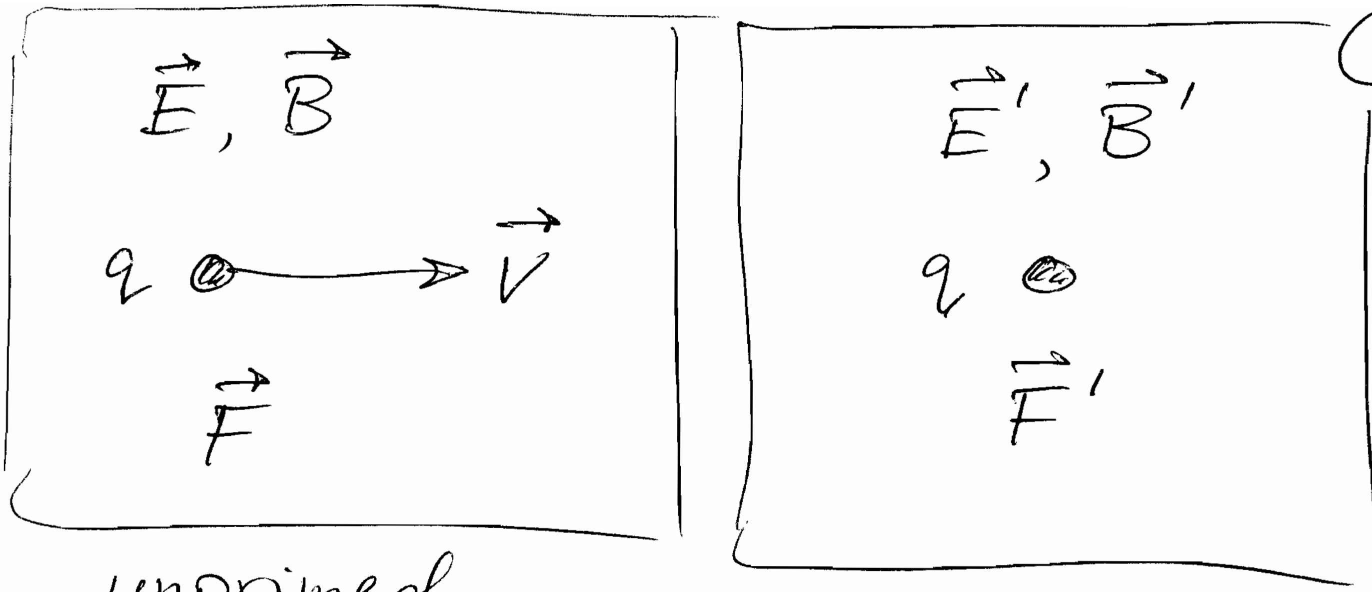
Lecture ~~32~~ 33

In this lecture we finish up some important unfinished business: the general Lorentz transformation rules for the electric <u>and</u> magnetic field. We could, as before, go about it by looking at the same source (charge density <u>and</u> current density) in relatively moving frames and compare the $\vec{E}$ and $\vec{B}$ we would see. For variety we will use a different approach: the invariance of the force law for charged particles.



unprimed frame | primed frame

The frames are related by a boost with velocity vector $\vec{V}$, chosen so the charge is at rest in the primed frame. The invariance of the law of force implies,

$$\vec{F} = q\vec{E} + q\vec{V}\times\vec{B}\,, \quad \vec{F}' = q\vec{E}'$$

(since the particle velocity happens

to be zero in the primed frame). In both frames we decompose the fields into components parallel and perpendicular to the velocity vector. The forces decompose as follows:

$$\vec{F}_{\parallel} = q\vec{E}_{\parallel}$$

$$\vec{F}_{\perp} = q\vec{E}_{\perp} + q\vec{V}\times\vec{B}_{\perp}$$

$$\vec{F}_{\parallel}' = q\vec{E}_{\parallel}'$$

$$\vec{F}_{\perp}' = q\vec{E}_{\perp}'$$

We used the fact that $\vec{V}\times\vec{B}$ is perpendicular to $\vec{V}$ and

$\vec{V} \times \vec{B} = \vec{V} \times \vec{B}_\perp$ to simplify these equations. We now use the relations derived in lecture 24 that apply when the primed frame is the instantaneous rest frame of the particle:

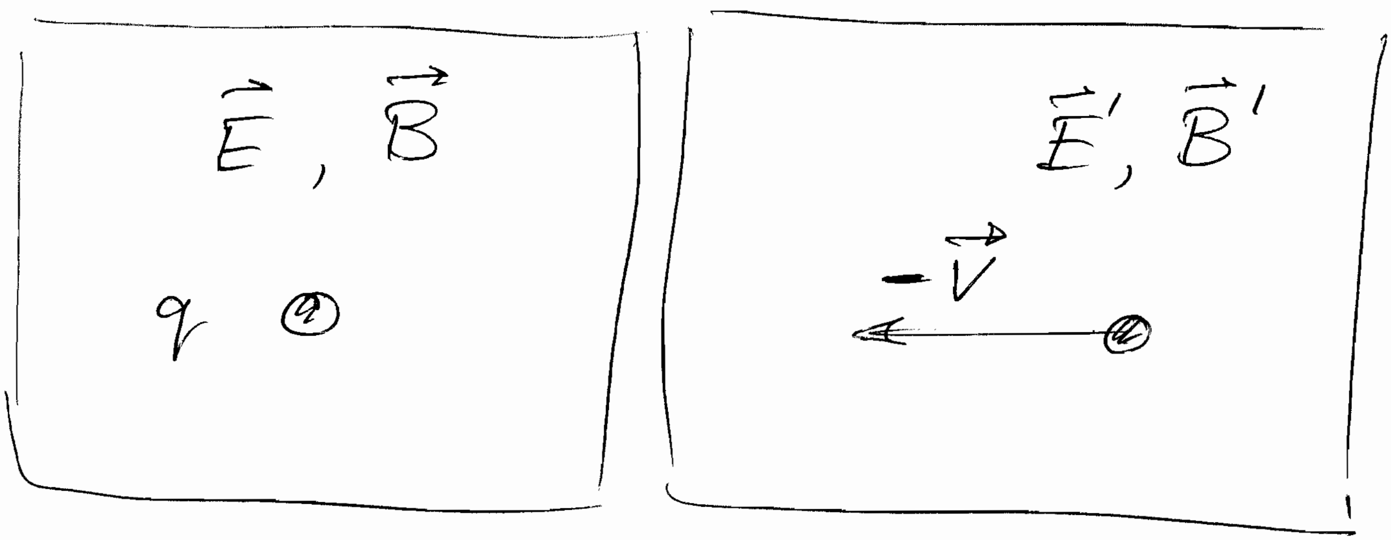
$$\vec{F}_{\parallel} = \vec{F}_{\parallel}' \ , \quad \vec{F}_\perp = \frac{1}{\gamma} \vec{F}_\perp'$$

$$\Rightarrow \quad \vec{E}_{\parallel} = \vec{E}_{\parallel}'$$

$$\Rightarrow \quad \gamma(\vec{E}_\perp + \vec{V} \times \vec{B}_\perp) = \vec{E}_\perp' \qquad (1)$$

We've seen these before for the special case where $\vec{B}_\perp = 0$.

Now let's look at two frames with the same fields but when the particle is in a different state of motion:



Now we have the following force equations:

$$\vec{F}_{\parallel} = q\vec{E}_{\parallel} \qquad \vec{F}_{\parallel}' = q\vec{E}_{\parallel}'$$

$$\vec{F}_{\perp} = q\vec{E}_{\perp} \qquad \vec{F}_{\perp}' = q\vec{E}_{\perp}' - q\vec{V}\times\vec{B}_{\perp}'$$

The frames are related by the same boost as before, but this time the particle rest frame is the unprimed frame, so $\frac{1}{\gamma}\vec{F}_\perp = \vec{F}'_\perp$:

$$\Rightarrow \quad \vec{E}_\parallel = \vec{E}'_\parallel$$

$$\Rightarrow \quad \vec{E}_\perp = \gamma\left(\vec{E}'_\perp - \vec{v}\times\vec{B}'_\perp\right) \quad (2)$$

This is all the physics we need (force law invariance), the rest is algebraic manipulation:

First substitute $\vec{E}'_\perp$ from equation (1) into (2):

$$\vec{E}_\perp = \gamma\left(\gamma\left(\vec{E}_\perp + \vec{v}\times\vec{B}_\perp\right) - \vec{v}\times\vec{B}'_\perp\right)$$

Next, take the cross product of both sides with $\vec{v}$ and use the identity

$$\vec{v}\times(\vec{v}\times\vec{B}_\perp) = -(\vec{v}\cdot\vec{v})\,\vec{B}_\perp$$

since $\vec{v}\cdot\vec{B}_\perp = 0$ (and similarly for the $\vec{B}'_\perp$ term):

$$(1-\gamma^2)\,\vec{v}\times\vec{E}_\perp + \gamma^2 v^2 \vec{B}_\perp = \gamma v^2 \vec{B}'_\perp$$

Finally, divide both sides by γv^2 and simplify:

$$\frac{\gamma^2-1}{\gamma v^2} = \frac{1}{\gamma}\left(\frac{\frac{1}{1-v^2/c^2}-1}{v^2}\right)$$

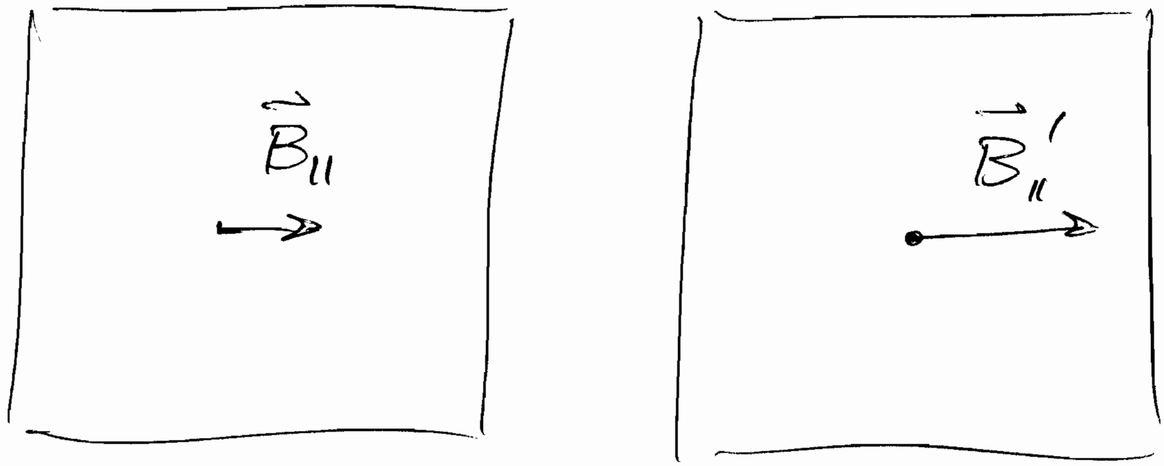
$$= \frac{1}{\gamma}\left(\frac{1/c^2}{1-v^2/c^2}\right) = \gamma/c^2 .$$

$$\Rightarrow \vec{B}_\perp' = \gamma\left(\vec{B}_\perp - \frac{1}{c^2}\vec{V}\times\vec{E}_\perp\right)$$

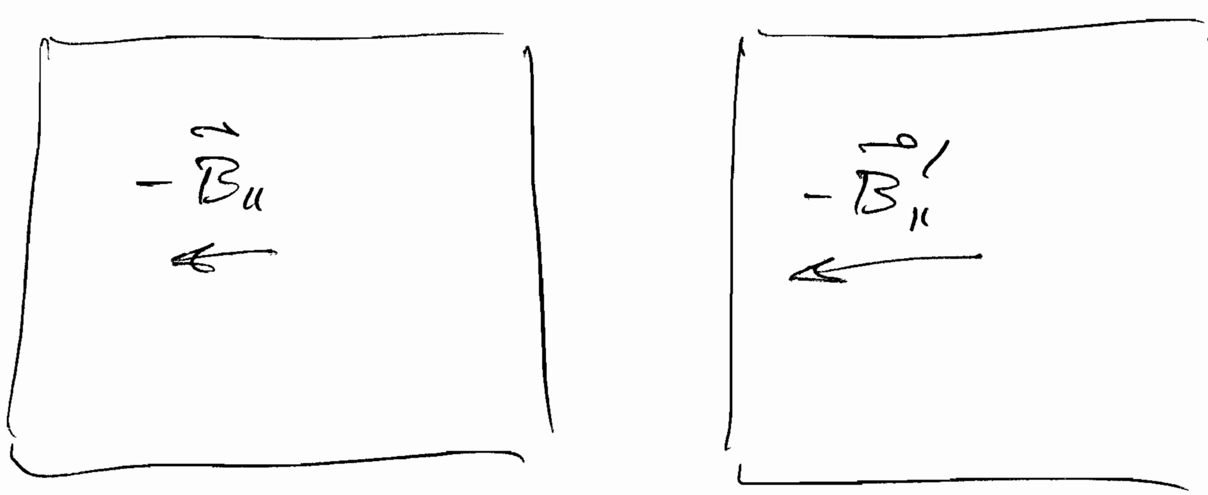
So far none of our transformations involved $\vec{B}_\parallel$ or $\vec{B}_\parallel'$. We can argue this case on the basis of symmetry and the superposition principles alone — we don't need the force law.

Suppose we only have a magnetic field in the unprimed frame. We'll orient it along the boost so $\vec{B}_\perp = 0$. By the previous transformations, $\vec{B}_\perp' = 0$, & $\vec{E}' = 0$ because $\vec{B}_\perp = 0$ and $\vec{E} = 0$.

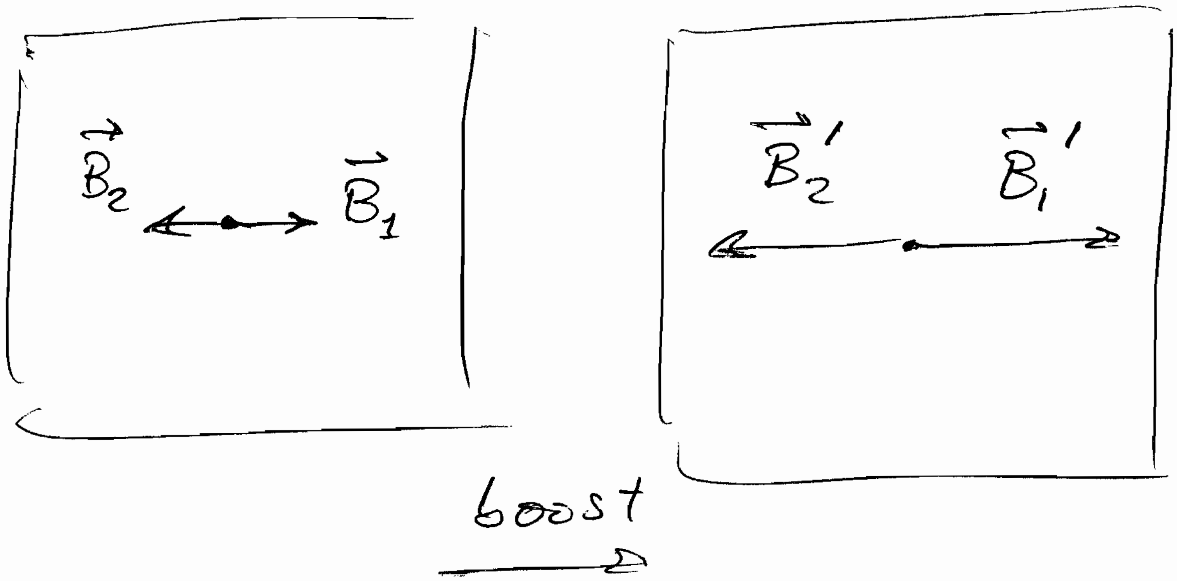
So in the primed frame we only have $\vec{B}_\parallel'$:



Now suppose we have $-\vec{B}_{\parallel}$ in the unprimed frame; then we better have $-\vec{B}'_{\parallel}$ in the primed frame since a superposition that gives zero field in one frame better give zero in any other frame:



But there is something seriously wrong with the following picture (summary of previous):



It suggests there is a preferential direction for boosting that ~~changes~~ increases the magnitude of $\vec{B}$. Since that is in conflict with the isotropy of space, we conclude that the magnitude is preserved, i.e.

$$\vec{B}_{\parallel} = \vec{B}_{\parallel}{}' .$$

We'll summarize the transformation rules here:

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} \qquad \vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

$$\vec{E}'_{\perp} = \gamma(\vec{E}_{\perp} + \vec{V} \times \vec{B}_{\perp})$$

$$\vec{B}'_{\perp} = \gamma\left(\vec{B}_{\perp} - \frac{1}{c^2} \vec{V} \times \vec{E}_{\perp}\right)$$