Lecture 31

In the previous lecture we showed that if $\vec{B}$ is constructed from a divergence-free vector potential $\vec{A}$, as $\vec{B} = \vec{\nabla} \times \vec{A}$, then automatically $\vec{\nabla} \cdot \vec{B} = 0$ and

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\nabla^2 \vec{A} .$$

By Ampère's law,

$$-\nabla^2 \vec{A} = \mu_0 \vec{j} .$$

This is essentially 3 independent Poisson equations:

$$-\nabla^2 A_x = \mu_0 j_x , \quad -\nabla^2 A_y = \mu_0 j_y ,$$

$$-\nabla^2 A_z = \mu_0 j_z .$$

These are solved by exactly the same integrals we had for the electrostatic Poisson equation:

$$\varphi \rightarrow A_x \ , \quad \rho/\varepsilon_0 \rightarrow \mu_0 \, j_x$$

$$A_x(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3\vec{r}\,' \, \frac{j_x(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

Combining all three components into one vector equation,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3\vec{r}\,' \, \frac{\vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \ .$$

Q: How can we be sure that this vector potential is divergence-free?

A: Recall that $\vec{A}$ satisfied

$-\nabla^2 \vec{A} = \mu_0 \vec{J}$; now take

divergence of both sides:

$$-\nabla^2 (\vec{\nabla} \cdot \vec{A}) = \mu_0 \vec{\nabla} \cdot \vec{J} \, .$$

Since $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = 0$ in

magneto statics, the divergence

of our $\vec{A}$ satisfies the Laplace

equation

$$\nabla^2 (\vec{\nabla} \cdot \vec{A}) = 0$$

with the usual boundary

conditions on Laplace's equation,

"φ" $= \vec{\nabla} \cdot \vec{A} \longrightarrow 0$ at infinity (which

our integral satisfies), uniqueness

implies "φ" $= \vec{\nabla}\cdot\vec{A} = 0$ (everywhere).

The final step in calculating $\vec{B}$ is to take the curl of $\vec{A}$.

For this we need two simple vector calculus identities:

$$(1) \quad \vec{\nabla}\times\left(f(\vec{r})\,\vec{V}\right) = (\vec{\nabla}f)\times\vec{V}$$

Here f is a scalar function and $\vec{V}$ a constant vector. You will check this in the homework.

$$(2) \quad \vec{\nabla}_{\vec{r}}\left(\frac{1}{|\vec{r}-\vec{r}\,'|}\right) = -\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

We've encountered this before, in connection with point charges.

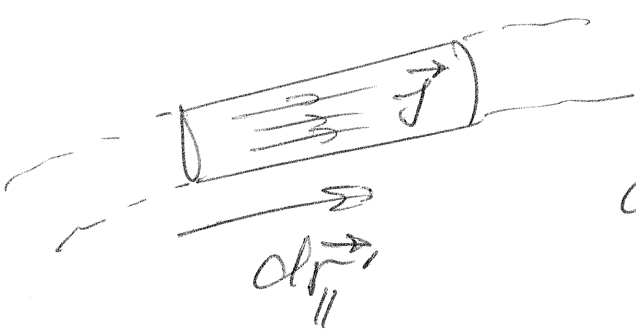
Using these identities when

taking the curl of our integral expression for $\vec{A}$, we obtain:

$$\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$$

$$= -\frac{\mu_0}{4\pi} \int d^3\vec{r}\,' \frac{(\vec{r}-\vec{r}\,') \times \vec{j}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

A special case of this, when $\vec{j}$ is confined to a thin wire carrying current I, is called the Biot-Savart law:



$$d^3\vec{r}\,' = d\vec{r}_{\parallel}'\, \underbrace{d^2\vec{r}_{\perp}'}_{\text{perp. to wire}}$$

$$\underbrace{d\vec{r}_{\parallel}' \; d^2 r_{\perp}' \; \vec{j}(\vec{r}')}_{I \text{ (independent of } \vec{r}'\text{)}}$$

$$\vec{B}(\vec{r}) = + \frac{\mu_0 I}{4\pi} \int \frac{d\vec{r}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}$$

(We dropped the parallel subscript on $\vec{r}'$ and changed the order of the cross-product.)