Lecture 30

In "magnetostatics" one is interested in the magnetic field created by a static distribution of currents.

Just as in electrostatics, one can work with the local equation

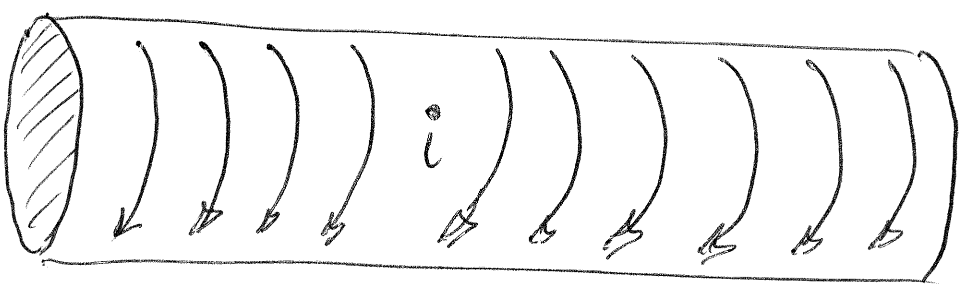
$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J},$$

or, when there is symmetry, their integral variants:

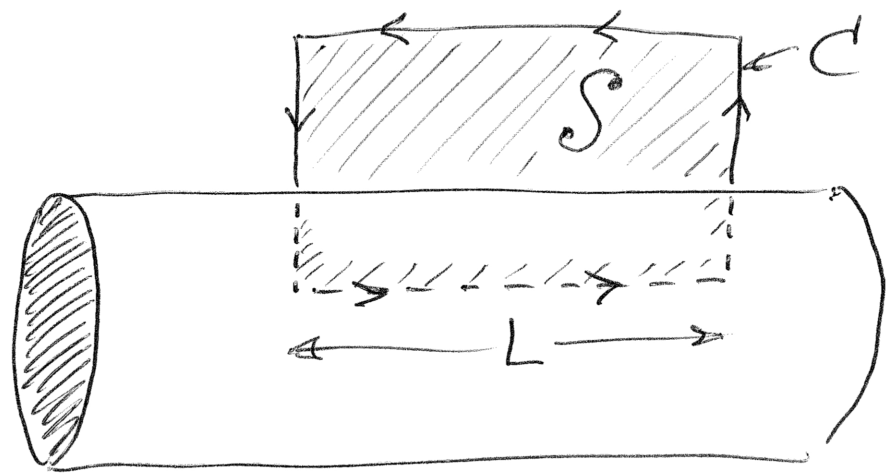
$$\oint_S \vec{E}\cdot d\vec{a} = \frac{Q_{enc.}}{\epsilon_0} \qquad \oint_C \vec{B}\cdot d\vec{r} = \mu_0 I_{enc.}$$

In This lecture we will illustrate both with examples. We start with the integral form of Ampere's law.

An important high-symmetry current source (apart from the straight wire) is shown below:



The current density is confined to a thin cylindrical surface, where it ~~flows~~ is uniform and flows everywhere perpendicular to the cylinder axis. It is characterized by a single number i, the current per unit length. Let's construct a rectangular surface S that intercepts the current as shown below:



The boundary curve C has been given an orientation so that

$$I_{enc} = i\,L$$

is positive if i is positive.

From the <u>symmetry</u> of the current distribution we immediately have two symmetry properties of the magnetic field:

(1) $\vec{B}$ is unchanged by translations parallel to the cylinder axis.

(2) $\vec{B}$ is unchanged by rotations about the cylinder axis.

Choosing a polar coordinate system with z-axis along the cylinder axis, these properties put the following restrictions on the form of $\vec{B}$:

$$\vec{B} = B_z(r)\,\hat{z} + B_r(r)\,\hat{r} + B_\phi(r)\,\hat{\phi}$$

The component B_r gives $\vec{B}$ a divergence, which is impossible. Consider a cylinderical surface S

co-axial with our current cylinder; then

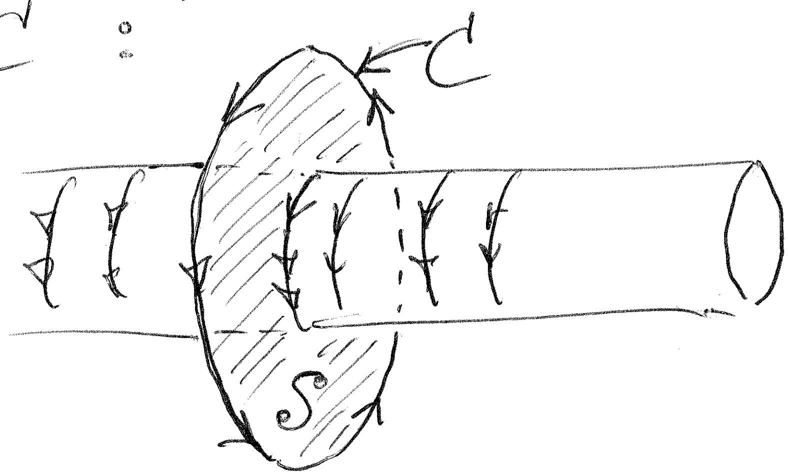
$$\oint_S \vec{B} \cdot d\vec{a} = 2\pi R L \, B_r(R) = 0 .$$

Here R and L are the radius and length of the cylinder S. This shows that $B_r(R) = 0$ for all R.

We next show that a non-zero B_ϕ is inconsistent with our current density. Take C to be a circle of radius R centered on the cylinder axis and perpendicular to it; then

$$\oint_C \vec{B} \cdot d\vec{r} = 2\pi R \, B_\phi(R) = \mu_0 I_{enc}$$

But $I_{enc} = 0$ for C, since our current density flows <u>within</u>, not through, the surface spanned by C:



The last equation therefore shows $B_\phi(R) = 0$ for all R.

We will now evaluate the circulation of $\vec{B} = B_z(r)\,\hat{z}$ for the curve C on page 3, for which $I_{enc} = i\,L$.

$$\oint_C \vec{B}\cdot d\vec{r} = B_z(R_1)L - B_z(R_2)L$$

$$= \mu_0 i L$$

$$\Longrightarrow \quad B_z(R_1) - B_z(R_2) = \mu_0 i$$

Here R_1 is the distance from the axis of the rectangle side inside the cylinder, and R_2 the distance of the opposite side, which lies outside the cylinder. Now consider another rectangular $\mathcal{C}$, with the same inside radius R_1 but a different outside radius R_3; then

$$B_z(R_1) - B_z(R_3) = \mu_0 i$$

Comparing the last two equations

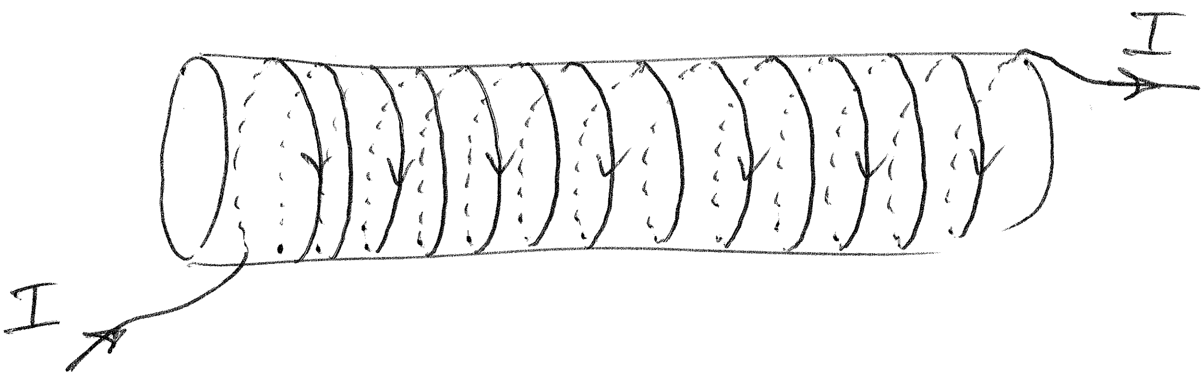
we obtain $B_z(R_2) = B_z(R_3)$

More generally, this shows that $B_z(R) = B_{out}$ for all outside R's, and similarly, $B_z(R) = B_{in}$ for all inside R's.

We therefore have reduced everything to two unknowns, with the relationship

$$B_{in} - B_{out} = \mu_0 i$$

Actually, any $\vec{B}$ with uniform z-component inside and outside and differing by $\mu_0 i$ is a valid solution. For the physical current source known as "the solenoid" one normally takes $B_{out} = 0$, so $B_{in} = \mu_0 i$. In a solenoid the azimuthal flow is achieved by winding a thin wire around a tube with very small "pitch":



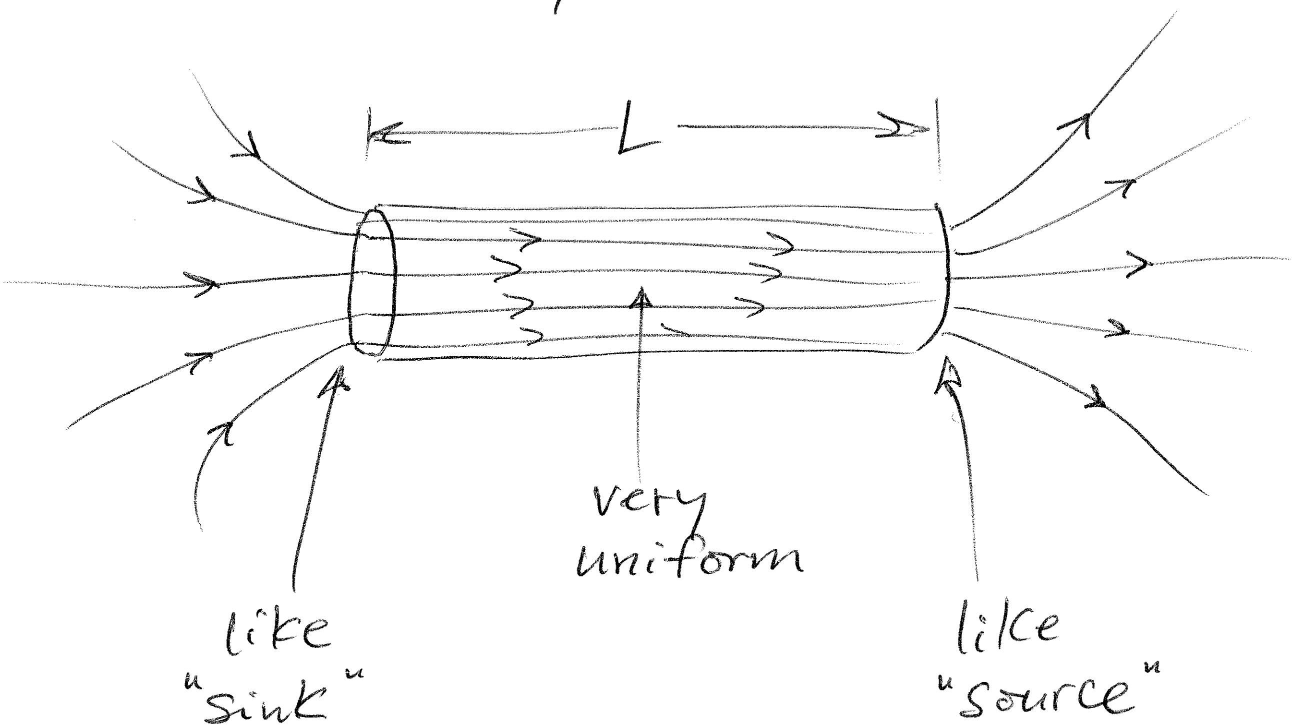
If there are n turns per unit length, and the wire carries current I, then

$$i = I \cdot n .$$

We choose $B_{out} = 0$ because the solenoid is a finite cylinder in a region with zero external magnetic field. So what we have learned is that the magnetic field of a long solenoid (much longer than its diameter) ~~the field~~ is zero on the outside and very uniform on the inside with magnitude

$$B_{in} = \mu_0 I n .$$

The field at the ends of a long solenoid is more complicated, but its qualitative behavior is exactly that of an electrostatic field, since $\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \times \vec{B} = 0$ (away from sources):



Very far from a finite solenoid ($R \gg L$) the magnetic field looks like the dipole field produced by a pair of monopoles (source and sink) separated by L.

When we don't have symmetry we must work with the local equations. Recall the steps in the case of electrostatics:

1) $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$

2) introduce potential: $-\vec{\nabla}\varphi = \vec{E}$

$$-\nabla^2\varphi = \rho/\epsilon_0 \quad \text{(Poisson)}$$

3) solve Poisson eqn:

$$\varphi(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

4) $\vec{E}(\vec{r}) = -\vec{\nabla}\varphi(\vec{r})$

Whereas a scalar potential φ

was "natural" for $\vec{E}$, since $\vec{\nabla}\times\vec{E}=0$ is automatically satisfied when $\vec{E}=-\vec{\nabla}\varphi$, it is not what we want in the case of $\vec{B}$ since $\vec{\nabla}\times\vec{B}\neq 0$ when there are currents. Instead, we realize that $\vec{\nabla}\cdot\vec{B}=0$ is <u>automatically satisfied</u> when

$$\vec{B}=\vec{\nabla}\times\vec{A}$$

i.e. whenever $\vec{B}$ happens to be the curl of some vector field $\vec{A}$ which we call the "vector potential".

The local Ampere's law takes the following form when written in terms of $\vec{A}$:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J}$$

home-work

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} .$$

This looks like the Poisson eqn. if can argue that $\vec{\nabla} \cdot \vec{A} = 0$ (or some other constant). We will exploit the fact that many $\vec{A}$'s describe the same $\vec{B}$ to select one for which $\vec{\nabla} \cdot \vec{A} = 0$.

Adding a gradient to $\vec{A}$ does not change $\vec{B}$:

Let $\vec{A}' = \vec{A} + \vec{\nabla} f$; then

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \underbrace{\vec{\nabla} \times (\vec{\nabla} f)}_{0} = \vec{\nabla} \times \vec{A}.$$

So the challenge is to find a suitable f such that $\vec{\nabla} \cdot \vec{A}' = 0$.

This was the subject of one of the homework problems, the first step of which was to realize that f satisfies the Poisson equation

$$\nabla^2 f = \underbrace{-\vec{\nabla} \cdot \vec{A}}_{\text{"source"}}$$

Solving for f with the given source, we can take its gradient and add it to $\vec{A}$ to obtain a divergence-free $\vec{A}'$