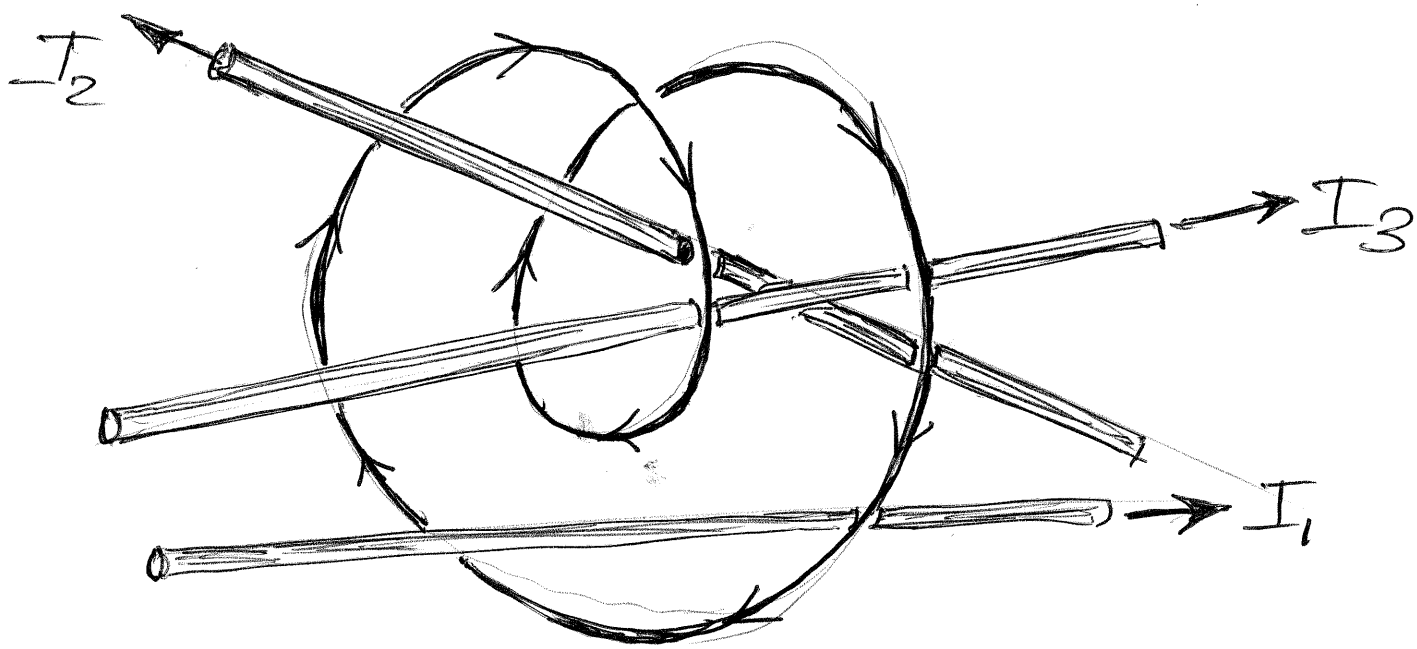
# Lecture 29

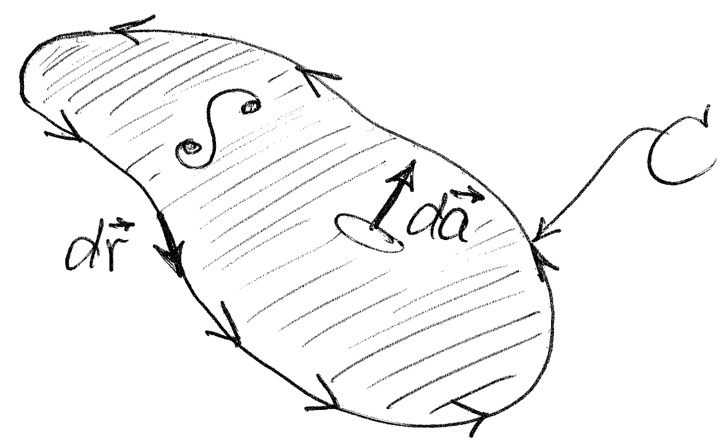
The notion of "enclosed current" is illustrated in the diagram below:




$$\oint_C \vec{B} \cdot d\vec{r} = \mu_0 I_{enc}$$

$$I_{enc} = I_1 - 2I_2 + 2I_3$$

A nice way to express the enclosed current $I_{enc}$ is with the help of a surface $\mathcal{S}$ that <u>spans</u> the closed curve $C$:




Both $C$ and $\mathcal{S}$ have a choice of <u>orientation</u>, and their orientations are related by the right-hand rule convention. The orientation of $\mathcal{S}$, specified by the direction of the surface elements $d\vec{a}$, is right-hand related to the sense in which $C$ circulates around the boundary of $\mathcal{S}$ (thumb $= d\vec{a}$, fingers curl in same sense as $C$).

A wire passing through $\mathcal{S}$ and carrying current $I$ from one side of $\mathcal{S}$ to the other in the same direction as $d\vec{a}$ corresponds to $I_{enc} = I$. The most general way for charge to flow through $\mathcal{S}$ is described by the current density vector field $\vec{j}$ (see lecture 13):

$$I_{enc} = \int_{\mathcal{S}} \vec{j} \cdot d\vec{a}$$

This formula would apply to arbitrarily complicated sets of straight wires passing through $\mathcal{S}$, for example. We will make the small leap of faith that the line integral of $\vec{B}$ will continue to reduce to this expression for $I_{enc}$ even when $\vec{J}$ is not a set of straight wires:

$$\oint_C \vec{B} \cdot d\vec{r} = \mu_0 \int_S \vec{J} \cdot d\vec{a}$$

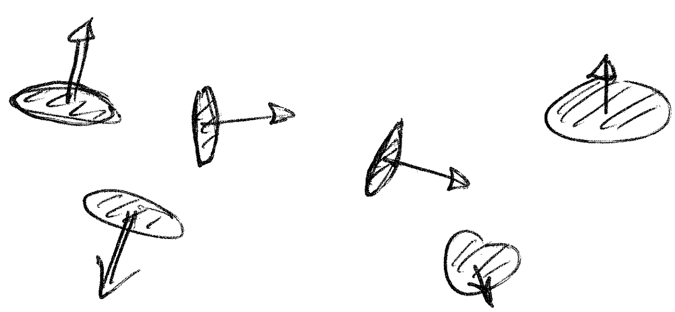
In a completely general setting (i.e. math), the same types of integrals and handedness conventions appear in Stoke's theorem:

$$\oint_C \vec{V} \cdot d\vec{r} = \int_S (\vec{\nabla} \times \vec{V}) \cdot d\vec{a}$$

Here $\vec{V}$ is an arbitrary vector field, $C$ an arbitrary closed curve, and $S$ a surface that spans $C$. Using Stoke's thm. we can replace our line integral of $\vec{B}$ with a surface integral:

$$\int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{a} = \mu_0 \int_S \vec{J} \cdot d\vec{a}$$

For this to be true for arbitrarily small surfaces $S$ and all orientations and positions, e.g.



the two vectors

$$\vec{\nabla} \times \vec{B} \quad \text{and} \quad \mu_0 \vec{J}$$

must be identical everywhere in space:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

This is the differential form of <u>Ampere's Law</u> — in a sense the magnetic counterpart of the differential Gauss's law.

A local, or differential law of exactly this type also applies to the electric field, although we never expressed it as such previously. Rather, we remarked that the electric field of a static set of charges creates a <u>conservative</u> force field, and this implies

$$\oint_C \vec{E} \cdot d\vec{r} = 0$$

for any closed curve $C$. We can rewrite this using Stoke's theorem as well,

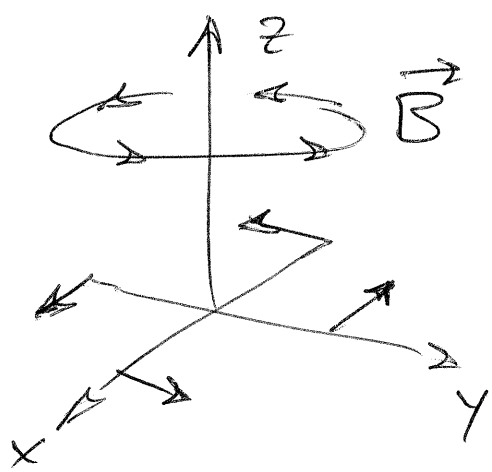
$$\int_S (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = 0,$$

where now $S$ is <u>any</u> surface in space. Since this includes arbitrarily small surfaces at any position and any orientation, this implies the local law

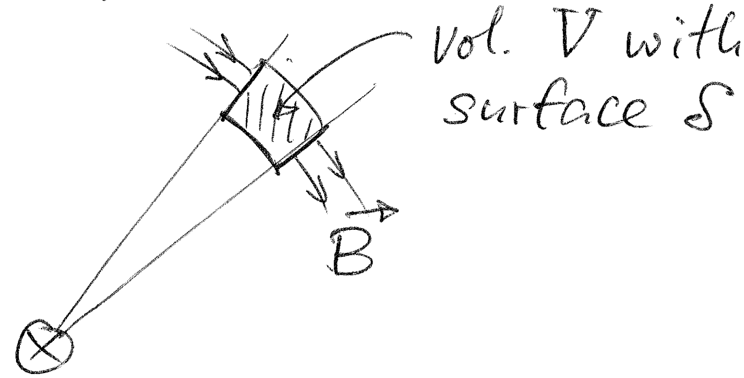
$$\vec{\nabla} \times \vec{E} = 0 .$$

While we're comparing $\vec{E}$ and $\vec{B}$, we should point out that the magnetic field of a straight wire has zero divergence. While it's possible to check this directly by evaluating $\vec{\nabla} \cdot \vec{B}$ using calculus and

$$\vec{B} = \frac{\mu_0 I}{2\pi} \frac{x\hat{y} - y\hat{x}}{x^2 + y^2}$$




for current $I$ flowing up the z-axis, it's immediately obvious when we consider the net flux of $\vec{B}$ into arbitrary small volumes:




$$\text{flux} = 0 = \oint_S \vec{B}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{B}\, d^3r$$

$$\Longrightarrow \quad \vec{\nabla}\cdot\vec{B} = 0$$

By superposition, the magnetic field created by multiple wires (or current density) will continue to be divergence-free.

Altogether we have four differential (local) laws for $\vec{E}$ and $\vec{B}$. These turn out to be a special case of the more general Maxwell equations when everything ($\vec{E}$, $\vec{B}$, $\rho$, $\vec{J}$) is <u>time-independent</u>:

$$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \qquad \vec{\nabla}\times\vec{E} = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}.$$

These, together with the force law,

$$\vec{F} = q\vec{E} + q\vec{V}\times\vec{B}$$

encapsulate most of what we know about electricity and magnetism at this point in the course.