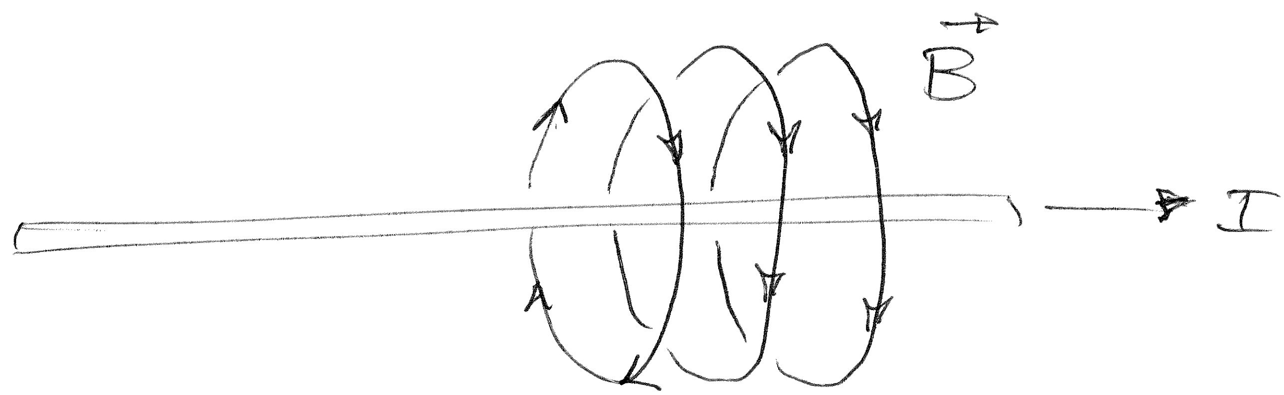
Lecture 28

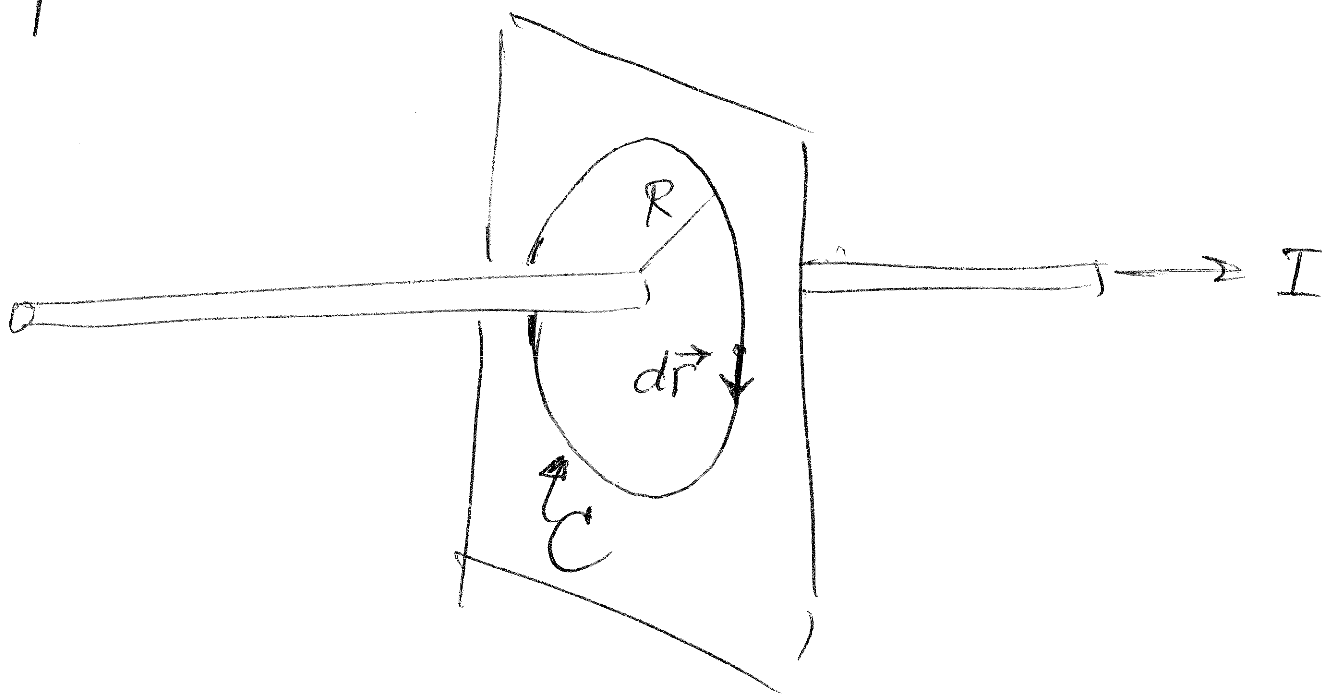
An infinite straight wire carrying current I is the source of magnetic field that encircles the wire in a right-handed sense (by definition) and has magnitude

$$B = \frac{\mu_0 I}{2\pi R}$$ decaying as the inverse distance R from the wire:



The line integral of $\vec{B}$ around a closed ~~path~~ curve C does not have any direct physical interpretation (as does the line integral of $\vec{E}$) but turns out to be a nice way to characterize the relationship between $\vec{B}$ and its sources — much like Gauss's law and $\vec{E}$.

Let's start with a curve C that's a <u>circle</u> of radius R, centered on the wire and in a plane perpendicular to the wire



$d\vec{r}$ = line element along C

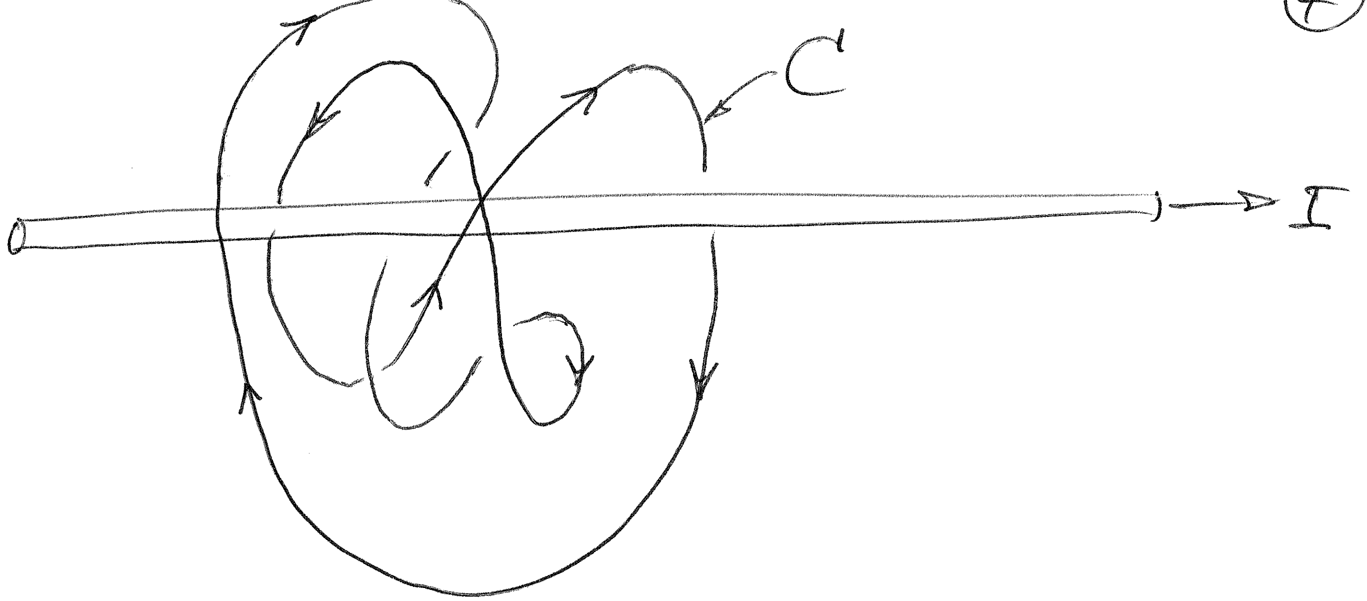
Because our choice of line element is everywhere parallel to $\vec{B}$,

$$\oint_C \vec{B}\cdot d\vec{r} = \oint_C \left(\frac{\mu_0 I}{2\pi R}\right)|d\vec{r}|$$

$$= \left(\frac{\mu_0 I}{2\pi R}\right) \underbrace{\oint_C |d\vec{r}|}_{2\pi R} = \mu_0 I$$

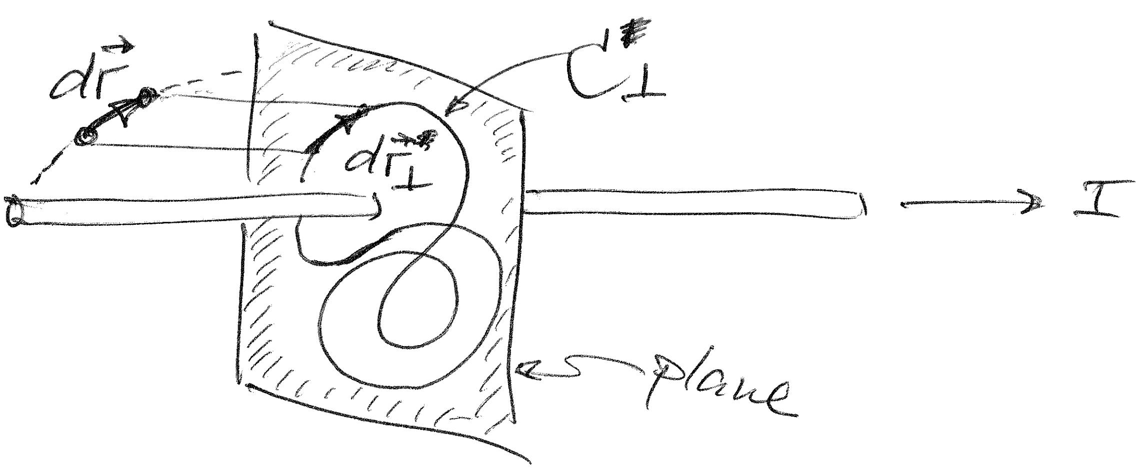
a result that is independent of the radius of C. We will now show that the line integral is independent of almost all properties of C, in the case when C is not just a simple circle (analogous to the insensitivity, in Gauss's law, of the shape of the surface S).

So let's imagine a complicated closed curve C:



The first thing we can do is project C into a plane perpendicular to the wire. If the resulting curve is $C_\perp^{\prime\#}$, then

$$\oint_C \vec{B}\cdot d\vec{r} = \oint_{C_\perp^{\prime\#}} \vec{B}\cdot d\vec{r}_\perp^{\,\prime\#}$$



The line integral is unchanged under this projection because:

(1) $\vec{B}$ is unchanged, when translated ~~along~~ parallel to the wire

(2) $\vec{B}$ has no component parallel to the wire:

$$\vec{B}\cdot d\vec{r} = \underbrace{\vec{B}_{\parallel}}_{0}\cdot d\vec{r}_{\parallel} + \underbrace{\vec{B}_{\perp}}_{\vec{B}}\cdot d\vec{r}_{\perp} = \vec{B}\cdot d\vec{r}_{\perp}$$

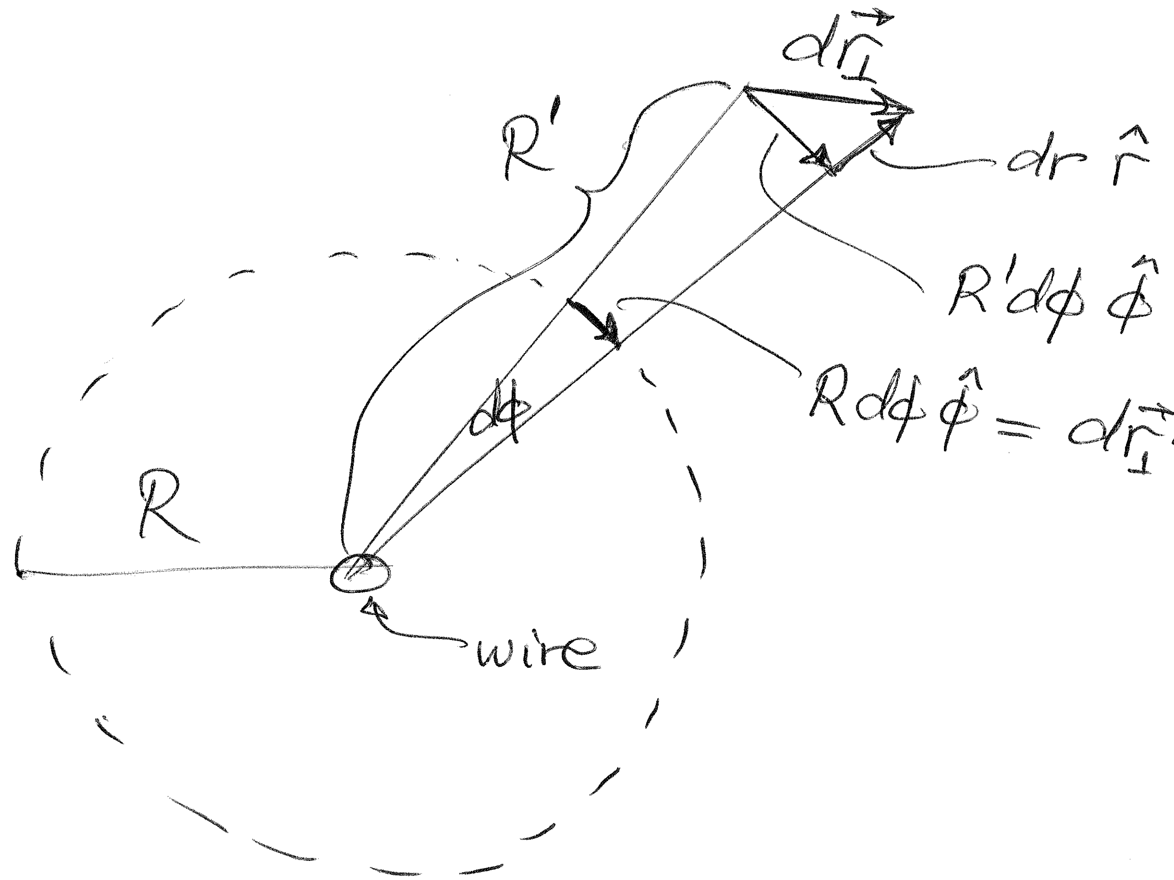
The second thing we can do is project each line element $d\vec{r}_{\perp}$ of $C_{\perp}$ onto a circle of radius R:

$$d\vec{r}_{\perp} = dr\,\hat{r} + R'\,d\phi\,\hat{\phi}$$

$$d\vec{r}_{\perp}{}' = 0 + R\,d\phi\,\hat{\phi}$$

(drawing on next page)

6

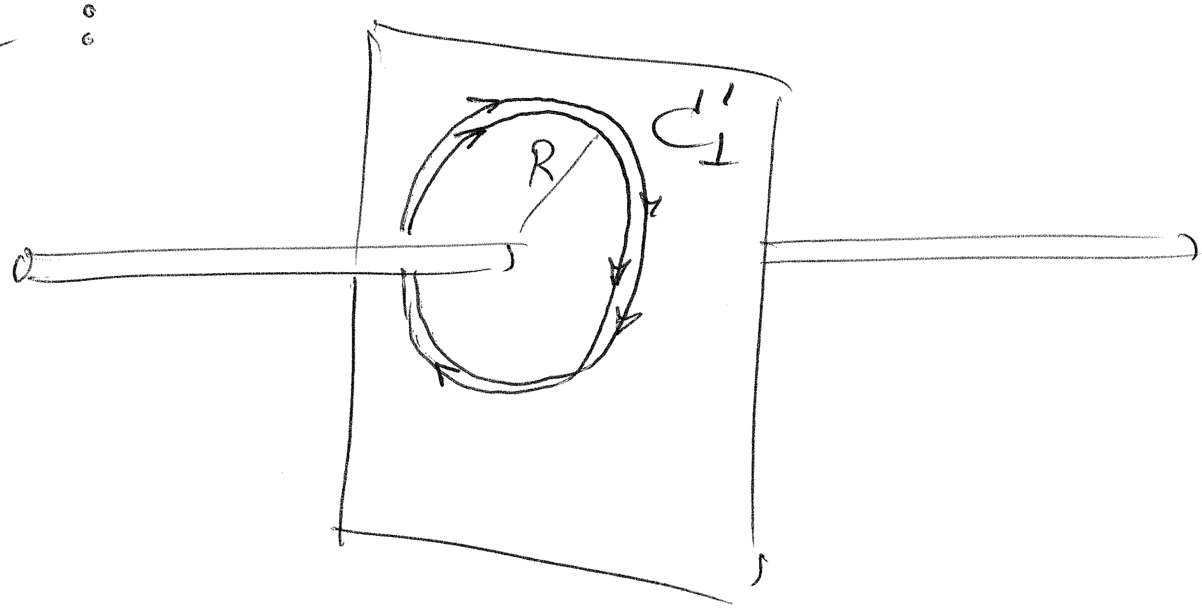


$$\vec{B}\cdot d\vec{r}_\perp = \vec{B}\cdot\left(R'd\phi\,\hat{\phi}\right) = \left(\frac{\mu_0 I}{2\pi R'}\right)R'd\phi$$

$$\parallel$$

$$\vec{B}\cdot d\vec{r}_\perp{}' \qquad = \left(\frac{\mu_0 I}{2\pi R}\right)R\,d\phi$$

So we only need to evaluate $\vec{B}$ on the circle of radius R and the line elements $d\vec{r}_\perp{}'$ all lie on this circle. When all the line elements are projected this way the result is a <u>circular</u> curve C_1' which wraps around the wire the same number of times, and in the same sense, as the original curve C:



But this is just the ~~line~~ very first line integral we performed, or multiple instances of it. For an arbitrary curve C we therefore have

$$\oint_C \vec{B} \cdot d\vec{r} = \mu_0 I \cdot N$$

$N =$ number of times C wraps around wire.

N can be zero, when C does not wrap around at all, or negative, when C wraps around in a left-handed sense. The "sense" of the wrapping is determined in part by the direction of I. Changing the direction of I changes the sense of the wrapping, and therefore the sign of N. Equivalently, we can change the sign of I (to reverse its direction) and this also changes the sign of $\mu_0 I N$.

Since the superposition principle applies to $\vec{E}$, it should also apply to $\vec{B}$. The magnetic field created by multiple wires carrying currents $I_1, I_2, \ldots$ should be the <u>sum</u> of their individual magnetic fields $\vec{B}_1, \vec{B}_2, \ldots$. Let $\vec{B}$ be this sum (the net magnetic field), then:

$$\oint_C \vec{B}\cdot d\vec{r} = \oint_C \vec{B}_1\cdot d\vec{r} + \oint_C \vec{B}_2\cdot d\vec{r} + \cdots$$

$$= \mu_0 I_1 N_1 + \mu_0 I_2 N_2 + \cdots$$

$$= \mu_0 I_{enc}$$

The "enclosed" current I_{enc} depends on both the currents flowing in the wires and the number of times C wraps around each of them.