Lecture 24

In addition to revealing how the electric field of a point charge becomes anisotropic when it is moving (causing radiation fields to form when the motion changes) the Lorentz transformation rule for the electric field is important for quite another thing entirely. Recall that conceptually the electric field began as "the local entity responsible for the _force_ on a charged particle". At the time we didn't concern ourselves with the state of motion of the particle

that experiences the force.
However, our assumption was
clearly that the charge was at rest
initially or at least moving slowly
(compared with c). But now that
we know how electric fields
transform, we can go into the
instantaneous rest frame of ~~a~~
even a rapidly moving charge
and work out the force there,
then transforming everything back
to the original frame.

So here's the scenario. In
the "lab" frame (unprimed) we have
a uniform $\vec{E}$ and a charge q
whose velocity at time $t=0$ is $\vec{V}$.

The momentum of the charge is $\vec{P}$ which we decompose as $\vec{P} = \vec{P}_{\parallel} + \vec{P}_{\perp}$, where $\vec{P}_{\parallel}$ is parallel to the instantaneous velocity $\vec{V}$ at $t = 0$. The electric field is decomposed according to the same axes defined by $\vec{V}$ into $\vec{E} = \vec{E}_{\parallel} + \vec{E}_{\perp}$. When we go into the "particle" frame (primed) we see a transformed electric field :

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} , \quad \vec{E}'_{\perp} = \gamma \vec{E}_{\perp} .$$

In the lab frame we would like to know

$$\frac{d\vec{P}_{\parallel}}{dt} = \cdots$$

$$\frac{d\vec{P}_{\perp}}{dt} = \cdots$$

at time $t=0$. We have omitted the arrow over $P_{\parallel}$ since its direction is defined by $\vec{v}$.

Using the Lorentz transformation we will transform the differentials in these expressions into the primed frame where ~~that~~ the particle is instantaneously at rest.

Recall that $\vec{P}$, together with the particle energy E/c, forms a 4-vector with transformation

$$P_{\parallel} = \gamma \left(P_{\parallel}' + \frac{v}{c^2} E' \right)$$

$$\vec{P}_{\perp} = \vec{P}_{\perp}' . \qquad\qquad v = |\vec{v}|$$

The same transformation applies to the differentials (differences of 4-vectors):

$$dP_\parallel = \gamma\left(dP_\parallel' + \frac{v}{c^2}dE'\right)$$

$$d\vec{P_\perp} = d\vec{P_\perp'}.$$

The energy and momentum magnitude satisfy the invariant-rest-mass relation

$$(E')^2 = (p'c)^2 + (mc^2)^2$$

from which we obtain

$$E'dE' = c^2 p'dp'$$

But in the instantaneous rest frame $p'=0$, so $dE'=0$. This corresponds, in non-relativistic language, to the fact that the power given to the particle in a force field vanishes at times

when the particle is at rest:

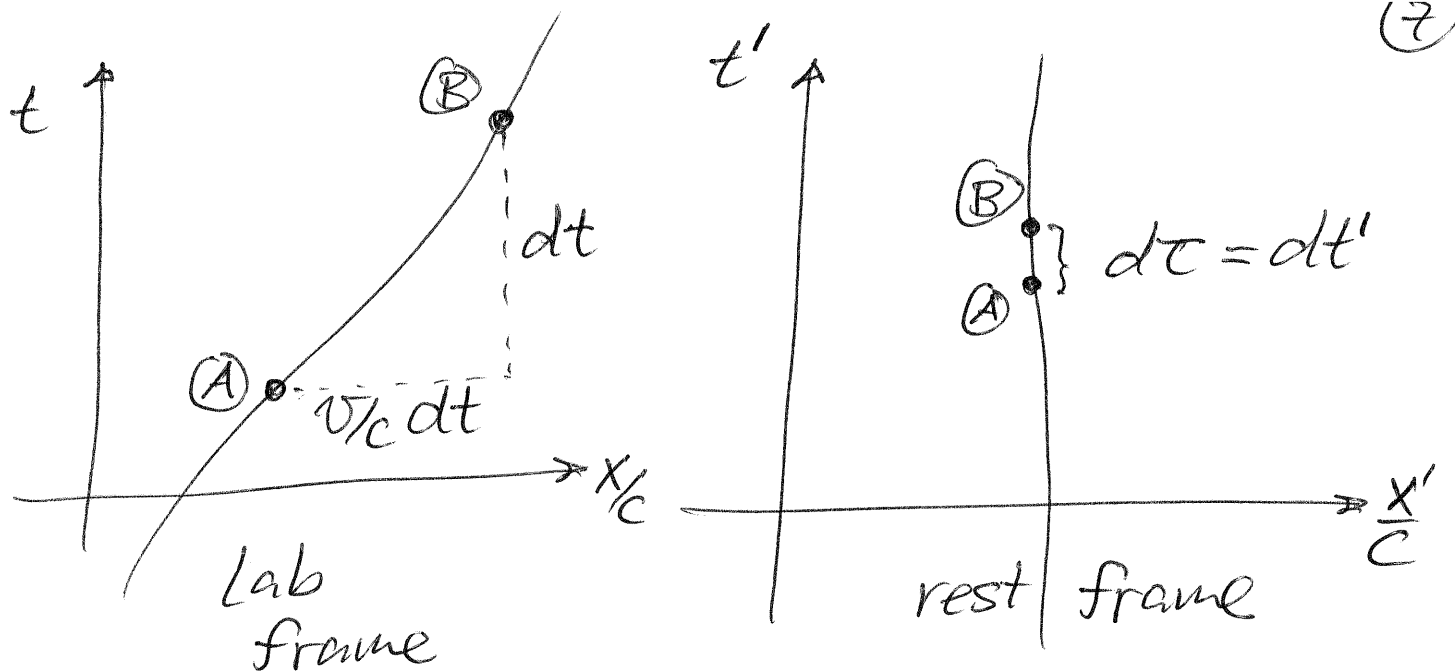$$\text{power} = \frac{d\mathcal{E}'}{dt'} = \vec{V}' \cdot \vec{F}$$

$$\vec{V}' = 0 \implies d\mathcal{E}' = 0 .$$

We thus get the simplification

$$dP_{\|} = \gamma \, dP_{\|}'$$

$$d\vec{P}_{\perp} = d\vec{P}_{\perp}' .$$

The differential $d\vec{P}$ corresponds to a momentum difference <u>between</u> <u>two events</u> on the particle's world line. Let's examine the relation between these (closely spaced) events in the two frames:

The proper time $d\tau$ between
events Ⓐ and Ⓑ is the same
in any frame and equals the
coordinate time dt' in the frame
where the particle is at rest :

$$d\tau = dt' = \sqrt{dt^2 - (v/c)^2 dt^2}$$

$$= \frac{dt}{\gamma}$$

We use this relation between dt'
and dt to transform our momentum

derivatives:

$$\frac{dP_{\parallel}}{dt} = \gamma \frac{dP_{\parallel}'}{dt} = \frac{dP_{\parallel}'}{dt'} = qE_{\parallel}' = qE_{\parallel}$$

$\underbrace{\phantom{\frac{dP_{\parallel}'}{dt'}}}$ force law in rest frame

$$\Rightarrow \quad \frac{dP_{\parallel}}{dt} = qE_{\parallel}$$

$$\frac{d\vec{P}_{\perp}}{dt} = \frac{d\vec{P}_{\perp}'}{dt} = \frac{1}{\gamma}\left(\frac{d\vec{P}_{\perp}'}{dt'}\right) \searrow \text{ force law}$$

$$= \frac{1}{\gamma}\left(q\vec{E}_{\perp}'\right) = q\vec{E}_{\perp}$$

$$\Rightarrow \quad \frac{d\vec{P}_{\perp}}{dt} = q\vec{E}_{\perp}$$

The content of these two laws,

which can be combined into
a single vector equation,

$$\frac{d\vec{P}}{dt} = q\vec{E} ,$$

is actually quite simple. The
rate of momentum change, as
calculated in any frame, is just
q times the electric field in
that frame.