Lecture 22

When observers in two inertial frames — "primed" and "unprimed" — are describing the same scalar property (e.g. temperature) they will have different functions, related by

$$f(\vec{r}, t) = f'(\vec{r}', t') .$$

For example, if the space-time event with unprimed coordinates $(\vec{r}, t)$ is assigned the value "42" by the unprimed observer's function f, then the primed observer's function f' better assign the same value "42" to the same event which will have unprimed coordinates $(\vec{r}', t')$.

In the case of vectorial properties, such as the electric field, we have to be more careful because the components of the vector make reference to the coordinate frame in question. In the previous lecture we found for the electric field the components parallel and perpendicular to the relative motion transform differently. The electric field functions are therefore related by

$$\vec{E}'_{\parallel}(\vec{r}'; t') = \vec{E}_{\parallel}(\vec{r}, t)$$

$$\vec{E}'_{\perp}(\vec{r}'; t') = \gamma \vec{E}_{\perp}(\vec{r}, t)$$

Using these equations we will

determine the field of a
moving point charge q.

In the rest frame of the
charge, the charge is located at
$x=y=z=0$ for all t :

$$E_x = \frac{Kq}{r^3} x \quad , \quad E_y = \frac{Kq}{r^3} y \quad , \quad E_z = \frac{Kq}{r^3} z$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

(The unprimed electric field is written
as a function of the unprimed coor-
dinates and is independent of t.)

Let the primed observer have
velocity $v\hat{x}$ as seen in the unprimed
frame; then
$$x' = \gamma(x - vt)$$
$$t' = \gamma(t - \frac{v}{c^2}x)$$
$$y' = y, \quad z' = z$$

are primed observer's coordinates

and

$$X = \gamma(X' + \upsilon t')$$

$$t = \gamma(t' + \frac{\upsilon}{c^2} X').$$

Let's first transform E_x. Since this component is parallel to the motion,
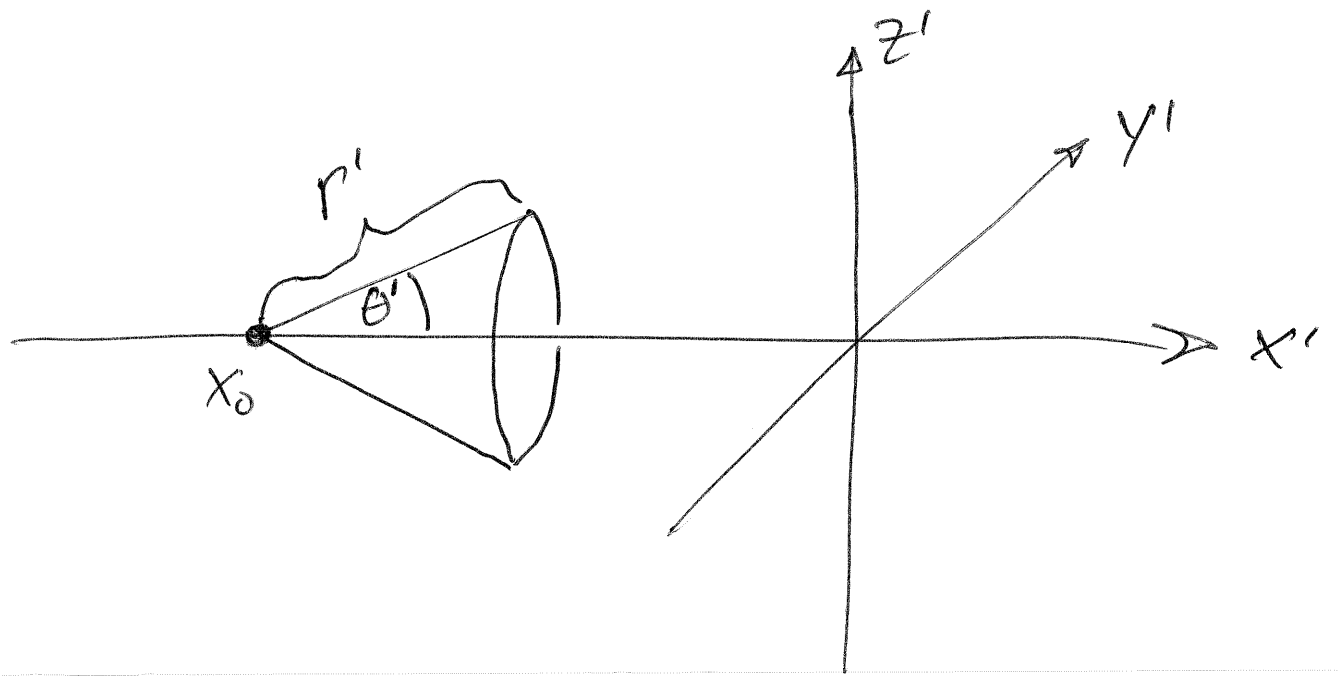
$$E_x' = E_x = \frac{Kq}{r^3} X.$$

We need to express this in terms of the primed coordinates:

$$\frac{X}{r^3} = \frac{\gamma(x' + \upsilon t')}{\left(\gamma^2(x' + \upsilon t')^2 + y'^2 + z'^2\right)^{3/2}}$$

To get a better grasp of this we will introduce spherical coordinates

in the primed frame. The
natural choice of origin is
the point $x' = -vt' = x_0$, $y' = 0$, $z' = 0$:

$$x' + vt' = x' - x_0 = r' \cos\theta'$$

$$\sqrt{y'^2 + z'^2} = r' \sin\theta'$$

$$\frac{x}{r^3} = \frac{(x' + vt')/\gamma^2}{\left((r'\cos\theta')^2 + \frac{1}{\gamma^2}(r'\sin\theta')^2 \right)^{3/2}}$$

The denominator can be
simplified:

$$r'^2 (1 - \sin^2\theta') + r'^2 \frac{1}{\gamma^2} \sin^2\theta'$$

$$= r'^2 \left(1 - \left(1 - \frac{1}{\gamma^2} \right) \sin^2\theta' \right)$$

$$= r'^2 \left(1 - \frac{v^2}{c^2} \sin^2\theta' \right)$$

$$E'_x = \frac{Kq}{\gamma^2 r'^3} \frac{x' - x_0}{\left(1 - \frac{v^2}{c^2} \sin^2\theta' \right)^{3/2}}$$

The y and z components of E' transform similarly, both being perpendicular to the motion:

$$E'_y = \gamma E_y = \frac{Kq}{r^3} \gamma Y = Kq \frac{\gamma Y'}{r^3}$$

$$= \frac{Kq}{\gamma^2 r'^3} \frac{Y'}{\left(1 - \frac{v^2}{c^2} \sin^2\theta' \right)^{3/2}}$$

$$E_z' = \frac{Kq}{\gamma^2 r'^3} \frac{z'}{(\ldots)^{3/2}}$$

$$(E_x', E_y', E_z') = \frac{Kq}{\gamma^2 r'^3 (\ldots)^{3/2}} (x'-x_0, y', z')$$

$$(x'-x_0, y', z') = r' \hat{r}'$$

Here $\hat{r}'$ is the unit vector that points away from the origin $(x_0, 0, 0)$. Summarizing:

$$\vec{E}' = \frac{Kq}{\gamma^2 r'^2} \frac{\hat{r}'}{(1 - v^2/c^2 \sin^2\theta')^{3/2}}$$