Physics 3318: Analytical Mechanics **Spring 2017**

Lecture 2: January 30

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2.1 Additivity of angular velocity

As you learned in freshman physics, when the river has velocity $\mathbf{v}_1$, and the kid swimming has velocity $\mathbf{v}_2$ relative to the river, then the kid's velocity relative to the shore is $\mathbf{v}_1 + \mathbf{v}_2$. This is a simple consequence of the additivity of translations. If the position of a floating ball relative to the shore is $\mathbf{r}_1$, and the position of the kid relative to the ball is $\mathbf{r}_2$, then the position of the kid relative to the shore is $\mathbf{r}_1 + \mathbf{r}_2$. Take the time derivative of this and you get the addition of relative velocity rule.

Rotations are not that simple: they are not combined by addition. As a physical scenario, suppose there is a pendulum mounted on a merry-go-round. During time Δt the pendulum (as body) has rotated about a horizontal axis relative to the merry-go-round (as "space") by U_1. But the merry-go-round has been rotating during this time, so we have to take the result of the first rotation and apply another rotation U_2, now relative to the fixed earth (true space). The net transformation of coordinates is therefore given by the (non-additive) product of rotations

$$U_{21} = U_2 U_1. \tag{2.1}$$

Fortunately, additivity still applies to angular velocity vectors. To see this, have the body and space frames coincide at time $t = 0$. We can do this because the body frame basis vectors are arbitrary, as long as we fix them once we've made our choice. As we learned in lecture 1, $\dot{U}_1 = A_1 U_1$, and therefore

$$U_1(\Delta t) \approx U_1(0) + \Delta t A_1(0) U_1(0) \tag{2.2}$$

is valid when Δt is small. Since the two frames coincide at $t = 0$, $U_1(0) = \mathbb{1}$ and we have

$$U_1(\Delta t) \approx \mathbb{1} + \Delta t A_1(0). \tag{2.3}$$

Here $A_1(0)$ is the antisymmetric matrix parametrized by the angular velocity vector $\boldsymbol{\omega}_1$ of the pendulum relative to the merry-go-round at time $t = 0$. By exactly the same argument

$$U_2(\Delta t) \approx \mathbb{1} + \Delta t A_2(0), \tag{2.4}$$

but where now $A_2(0)$ corresponds to the angular velocity vector $\boldsymbol{\omega}_2$ of the merry-go-round relative to the earth. Taking the product

$$U_{21}(\Delta t) = U_2(\Delta t) U_1(\Delta t) \approx \mathbb{1} + \Delta t \left(A_2(0) + A_1(0)\right) + O(\Delta t^2), \tag{2.5}$$

and comparing with the equation $\dot{U}_{21} = A_{21} U_{21}$, we see that

$$A_{21}(0) = A_2(0) + A_1(0). \tag{2.6}$$

Additivity of the antisymmetric matrix A — the time-rate-of-change of U — implies additivity of the associated angular velocity vectors:

$$\boldsymbol{\omega}_{21} = \boldsymbol{\omega}_2 + \boldsymbol{\omega}_1. \tag{2.7}$$

Think of this as a statement about three frames, just like the kid swimming in the river. When frames 0 and 1 are related by angular velocity $\boldsymbol{\omega}_1$, and frames 1 and 2 by angular velocity $\boldsymbol{\omega}_2$, the upshot is that frames 0 and 2 are then related by their sum.

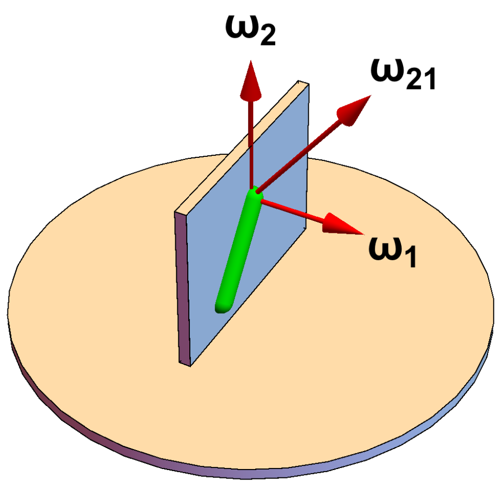

Figure 2.1: Sum of angular velocities applied to a pendulum (green) fixed to a rotating merry-go-round. The pendulum is constrained to swing in a vertical plane (shown) that is fixed to the merry-go-round. At this instant of time the angular velocity of the pendulum, relative to the plane, is $\boldsymbol{\omega}_1$. The plane has angular velocity $\boldsymbol{\omega}_2$ relative to the earth because it is fixed to the merry-go-round. The net angular velocity of the pendulum relative to the earth, at this instant of time, is $\boldsymbol{\omega}_{21} = \boldsymbol{\omega}_2 + \boldsymbol{\omega}_1$.

2.2 Fictitious forces

The body frame basis vectors are special cases of vectors fixed to the body whose time derivatives we worked out in lecture 1:

$$\dot{\hat{\mathbf{x}}}' = \boldsymbol{\omega} \times \hat{\mathbf{x}}' \qquad \dot{\hat{\mathbf{y}}}' = \boldsymbol{\omega} \times \hat{\mathbf{y}}' \qquad \dot{\hat{\mathbf{z}}}' = \boldsymbol{\omega} \times \hat{\mathbf{z}}'. \tag{2.8}$$

Now consider an arbitrary vector

$$\mathbf{a} = a'_x \hat{\mathbf{x}}' + a'_y \hat{\mathbf{y}}' + a'_z \hat{\mathbf{z}}', \tag{2.9}$$

where we allow the body frame components $a'_x(t)$, etc. to change with time. For example, if $\mathbf{a} = \mathbf{r}$ were a position it could be moving relative to the body. Let's compute the time derivative of this vector:

$$\begin{aligned}\dot{\mathbf{a}} &= \dot{a}'_x\hat{\mathbf{x}}' + \dot{a}'_y\hat{\mathbf{y}}' + \dot{a}'_z\hat{\mathbf{z}}' \quad + \quad a'_x\dot{\hat{\mathbf{x}}}' + a'_y\dot{\hat{\mathbf{y}}}' + a'_z\dot{\hat{\mathbf{z}}}' \\ &= \mathring{\mathbf{a}} + \boldsymbol{\omega} \times (a'_x\hat{\mathbf{x}}' + a'_y\hat{\mathbf{y}}' + a'_z\hat{\mathbf{z}}') \\ &= \mathring{\mathbf{a}} + \boldsymbol{\omega} \times \mathbf{a}. \end{aligned} \tag{2.10}$$

We'll use an open circle above vectors to denote a frame-based time derivative[1]:

$$\mathring{\mathbf{a}} = \text{time derivative of } \mathbf{a} \text{ “as seen in the body frame”}.$$

Equation (2.10) applies to *any* vector whose components we choose to express in terms of the rotating basis vectors $\hat{\mathbf{x}}'$, $\hat{\mathbf{y}}'$ and $\hat{\mathbf{z}}'$. For example, when applied to $\mathbf{a} = \boldsymbol{\omega}$ we get

$$\dot{\boldsymbol{\omega}} = \mathring{\boldsymbol{\omega}}. \tag{2.11}$$

The case we will be most interested in is where our general vector $\mathbf{a}$ is the velocity vector

$$\mathbf{a} = \dot{\mathbf{r}} \tag{2.12}$$

$$= \mathring{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}. \tag{2.13}$$

Applying equation (2.10) to this vector we get

$$\dot{\mathbf{a}} = (\mathring{\mathring{\mathbf{r}}} + \boldsymbol{\omega} \times \mathring{\mathbf{r}}) + \mathring{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\mathring{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}) \tag{2.14}$$

$$\ddot{\mathbf{r}} = \mathring{\mathring{\mathbf{r}}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \mathring{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r}. \tag{2.15}$$

The point of the kinematical relationship above is to relate the true acceleration of a particle, $\ddot{\mathbf{r}}$, to the apparent acceleration "as seen in the body frame", $\mathring{\mathring{\mathbf{r}}}$. Say the particle has mass m. The true force acting on the particle is

$$\mathbf{F}_{\text{true}} = m\ddot{\mathbf{r}}, \tag{2.16}$$

while the force that "explains" the acceleration seen in the body frame is

$$\mathbf{F}_{\text{body}} = m\mathring{\mathring{\mathbf{r}}}. \tag{2.17}$$

Now if we insist on making sense of motion in the body frame — knowing full well that it is not an inertial frame — we can do so by introducing fictitious forces to make up the difference:

$$\mathbf{F}_{\text{body}} = \mathbf{F}_{\text{true}} + \mathbf{F}_{\text{fict}}, \tag{2.18}$$

$$\mathbf{F}_{\text{fict}}/m = -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2\boldsymbol{\omega} \times \mathring{\mathbf{r}} - \dot{\boldsymbol{\omega}} \times \mathbf{r}. \tag{2.19}$$

The first two terms in the fictitious force have special names. The centrifugal force

$$\mathbf{F}_{\text{cent}}/m = -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{2.20}$$

scales as ω^2 and depends on the position of the particle relative to the origin (axis of rotation). The Coriolis force

$$\mathbf{F}_{\text{cor}}/m = -2\boldsymbol{\omega} \times \mathring{\mathbf{r}} \tag{2.21}$$

[1] Veit Elser retains full intellectual property rights to this ground breaking notation.

scales as ω^1 and applies only when the particle has a nonzero velocity ($\mathring{\mathbf{r}} \neq 0$) in the body frame. The third term in the fictitious force is zero or very small in many situations, such as Earth-bound observations, where the angular velocity vector is constant or nearly so.

Question: Explain the relationship, shared by all three fictitious forces, between the power of ω and the number of time derivatives.

Question: Consider the most commonly encountered situation, where $\boldsymbol{\omega}$ is constant and nonzero. One of the fictitious forces violates time-reversal symmetry — which one?