Lecture 17

We will soon study a third circuit component, the "capacitor". Because capacitors are often described as storing electrical energy (simply through an electric field rather than chemically, as in batteries) we need to review the mathematics of electrical energy.

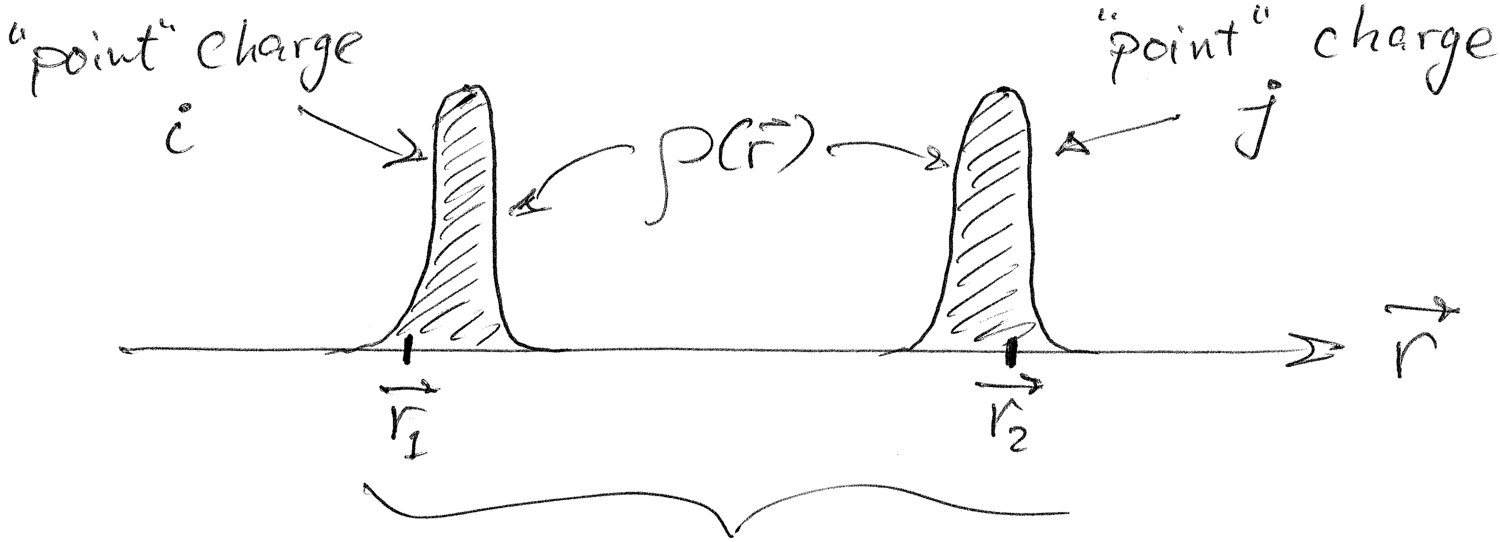
Our starting point is the Coulomb energy of a system of point charges, which we arrived at with the work-energy theorem applied to the Coulomb force:

$$U = \frac{K}{2} \sum_{i \neq j} \frac{q_i \, q_j}{|\vec{r}_i - \vec{r}_j|}$$

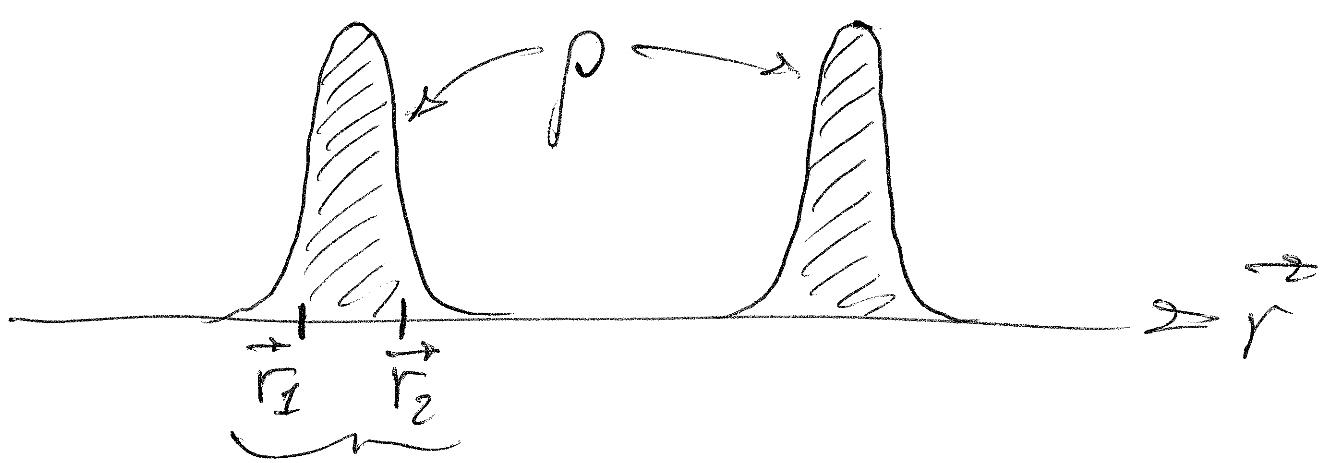
It will be easier to manipulate this expression if we work with charge distributions instead of point charges:

$$U = \frac{K}{2} \int d^3r_1 \int d^3r_2 \; \frac{\rho(\vec{r}_1)\, \rho(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|}$$

Notice that we did not add a restriction $\vec{r}_1 \neq \vec{r}_2$ to take the place of $i \neq j$. To better understand this, consider two point charges that have been slightly spread out — This would be done in all three dimensions but the sketches below are in 1D:



when $\vec{r}_1$ & $\vec{r}_2$ are like this, we get an $i \neq j$ contribution



when $\vec{r}_1$ & $\vec{r}_2$ are like this, we get a new "self-energy" contribution $(i=j)$

So $\tilde{U}$ contains self-energy that U did not. We can deal with that by asserting that every point charge is always spread-out the same way, so the two energies $\tilde{U}$ and U always differ by a fixed amount and it makes no difference which we use.

Recall the electric potential of a point charge q_2 located at $\vec{r}_2$:

$$\varphi(\vec{r}) = K \frac{q_2}{|\vec{r}-\vec{r}_2|} \quad (1 \text{ point charge})$$

We superpose these potentials

to get the potential of a distribution of charges:

$$\varphi(\vec{r}) = K \int d^3 r_2 \frac{\rho(\vec{r}_2)}{|\vec{r} - \vec{r}_2|} \quad \left(\begin{array}{c}\text{distr. of} \\ \text{charges}\end{array}\right)$$

Notice that $\varphi(\vec{r}_1)$ is exactly what we have in our $\tilde{U}$ double integral:

$$\tilde{U} = \frac{1}{2} \int d^3 r_1 \, \rho(\vec{r}_1) \, \varphi(\vec{r}_1) \longrightarrow U$$

This is our first, calculus-style energy expression. In the future we will ignore the constant (and uninteresting) difference with U.

Notice that we can drop the subscript 1 since $\vec{r}_1$ is an arbitrary integration variable name.

$$\mathcal{U} = \frac{1}{2}\int d^3r\, \rho(\vec{r})\, \varphi(\vec{r})$$

To get the second form, we first use the differential Gauss's law to make the substitution

$$\rho(\vec{r}) = \epsilon_0\, \vec{\nabla}\cdot\vec{E}(\vec{r})$$

$$\mathcal{U} = \frac{\epsilon_0}{2}\int d^3r\, \vec{\nabla}\cdot\vec{E}(\vec{r})\, \varphi(\vec{r}) .$$

Next we use the vector-calculus identity you proved in the homework:

$$\vec{\nabla}\cdot(\vec{E}\varphi) = (\vec{\nabla}\cdot\vec{E})\varphi + \vec{E}\cdot\vec{\nabla}\varphi$$

$$\Rightarrow U = \frac{\epsilon_0}{2}\int d^3r\,\Big(\underbrace{\vec{\nabla}\cdot(\vec{E}\varphi)}_{\text{I}} - \underbrace{\vec{E}\cdot\vec{\nabla}\varphi}_{\text{II}}\Big)$$

We will show that I is zero using the divergence theorem:

$$\int_V d^3r\,\vec{\nabla}\cdot(\vec{E}\varphi) = \oint_{\text{boundary}(V)} (\vec{E}\varphi)\cdot d\vec{a}$$

Let V be a sphere of radius R, then as $R \to \infty$ V becomes all of space and the boundary of V has surface area proportional to R^2. But in a bounded-in-size distribution of charge, $|\vec{E}| \sim 1/R^2$ (or faster if net charge is zero) and $\varphi \sim 1/R$. Thus the surface integral vanishes as $R \to \infty$. We therefore have

$$U = -\frac{\epsilon_0}{2} \int d^3r \ \vec{E} \cdot \vec{\nabla} \varphi$$

The last step is to use the

relationship between φ and $\vec{E}$:

$$\vec{\nabla}\varphi = -\vec{E}$$

$$\Rightarrow \quad U = \frac{\epsilon_0}{2}\int d^3r \, |\vec{E}|^2$$

With this formula we have gone completely from an energy that makes reference only to charge, to an energy density

$$u = \frac{\epsilon_0}{2}|\vec{E}|^2$$

that depends entirely on electric field.