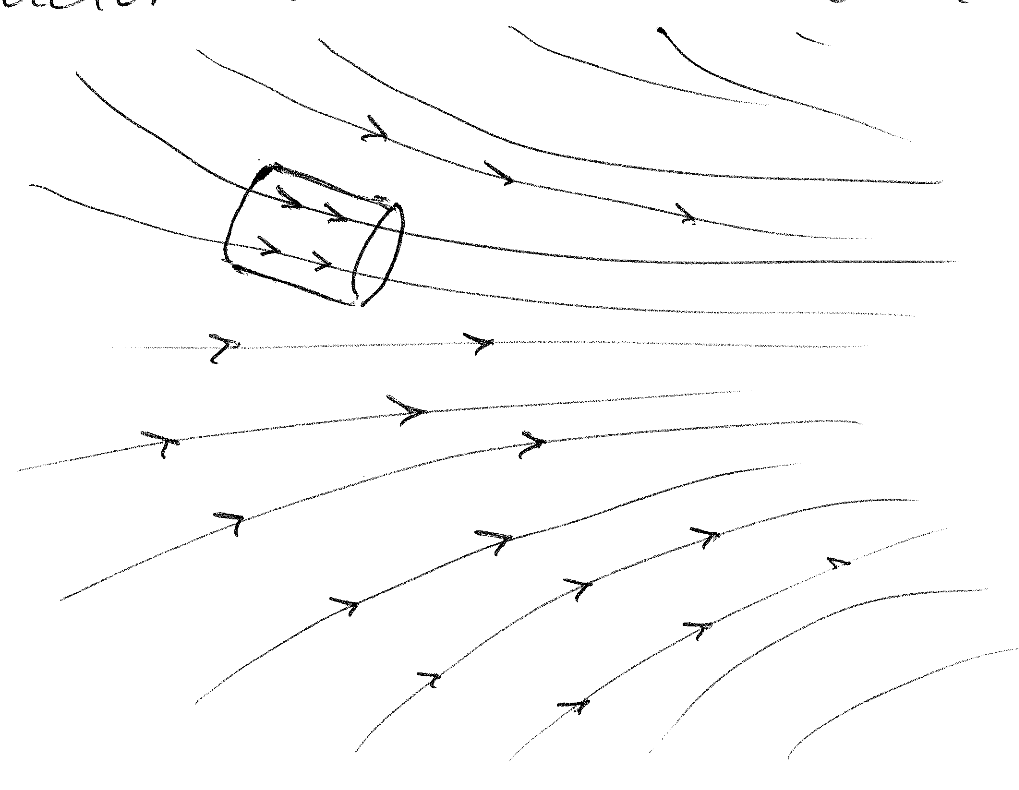
Lecture 13

Just as fluids and gases can have steady flows that are more varied than "uniform flow down a pipe" (last lecture) the same applies to free charge in a conductor. Let's think of the cylinder of uniform charge flow of last lecture as a small subvolume in a conductor with non-uniform flow:



The lines with arrows in the drawing represent either $\vec{E}$ or $\vec{v}_D$; they will continue to obey the proportionality $\vec{v}_D = \mu \vec{E}$.
In this non-uniform setting it makes no sense to define the current I (what area does it flow through?), instead we define the "current density" : $\vec{j}(\vec{r})$. The <u>direction</u> of this vector field is the same as that of $\vec{E}(\vec{r})$ or $\vec{v}_D(\vec{r})$ (when $q>0$). To specify its <u>magnitude</u> we consider an element of area $d\vec{a}$, exactly as we defined this for flux/Gauss's law, and the

definition

$$\vec{j} \cdot d\vec{a} = \text{flux of charge through surface element } d\vec{a}$$

To be more explicit, take $d\vec{a}$ to point parallel to $\vec{j}$ (same direction as $\vec{v}_D$ for positive q) ~~and~~ and have ~~area~~ ~~A~~ magnitude A (as in our cylinder geometry). Then the flux of charge through A is just the current I we worked out in the previous lecture:

$$|\vec{j}|\,|d\vec{a}| = I$$

$$|\vec{j}|\,A = nq\mu|\vec{E}|\,A$$

$$\Rightarrow \quad |\vec{J}| = nq\mu|\vec{E}|$$
$$= nq|\vec{v}_D|$$

A single equation that gives both direction and magnitude is

$$\vec{J} = nq\vec{v}_D$$

(the directions of $\vec{J}$ and $\vec{v}_D$ are opposite when $q<0$).

From $\vec{J}(\vec{r})$ we can determine the flow of charge through any surface S, even a closed one:

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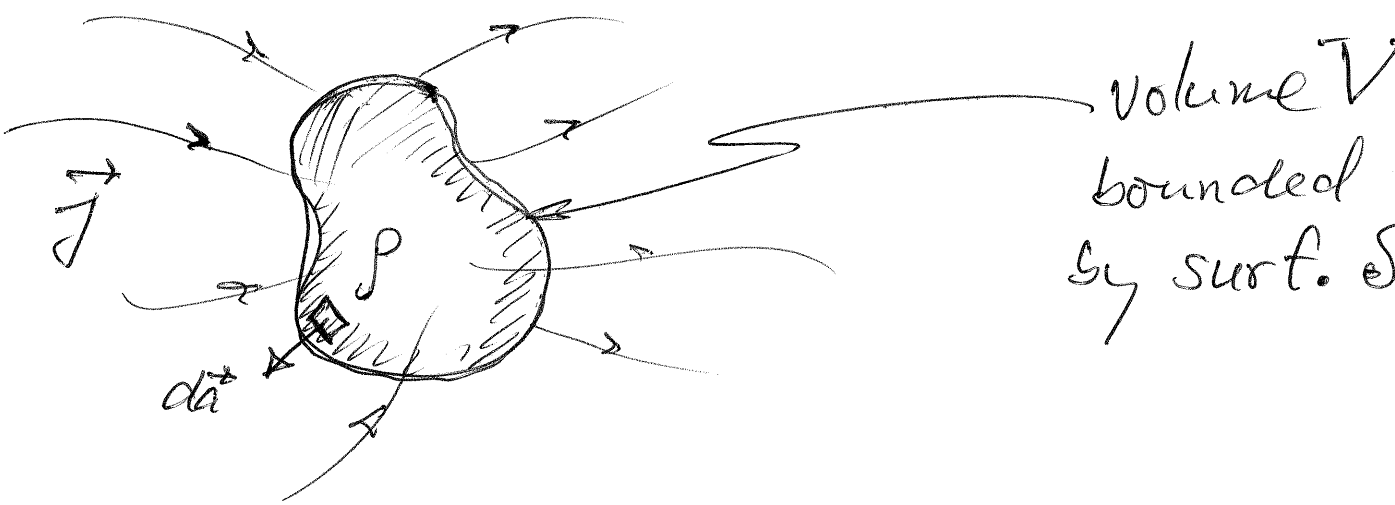
$$\oint_S \vec{J}\cdot d\vec{a} = \begin{pmatrix} \text{rate at which} \\ \text{charge flows } \cancel{\text{into}} \text{ out of} \\ \text{region bounded by } S \end{pmatrix}$$

$$= -\frac{\partial}{\partial t} Q_{enc}.$$

$$= -\frac{\partial}{\partial t}\left(\int_V \rho(\vec{r})\, d^3r \right)$$



Use the divergence theorem to transform the surface integral on the left-hand-side into a volume integral:

$$\int_V \vec{\nabla}\cdot\vec{J}\, d^3r = \int_V -\frac{\partial \rho(\vec{r})}{\partial t}\, d^3r$$

Taking V to be an arbitrary small volume (exactly as when deriving the differential form of Gauss's law) we find

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial \rho}{\partial t}$$

We'll refer to this as the "differential statement of charge conservation".

In steady-state flow

$\frac{\partial \rho}{\partial t} = 0$ by definition (in fact, $\rho = 0$ at all times in a conductor)

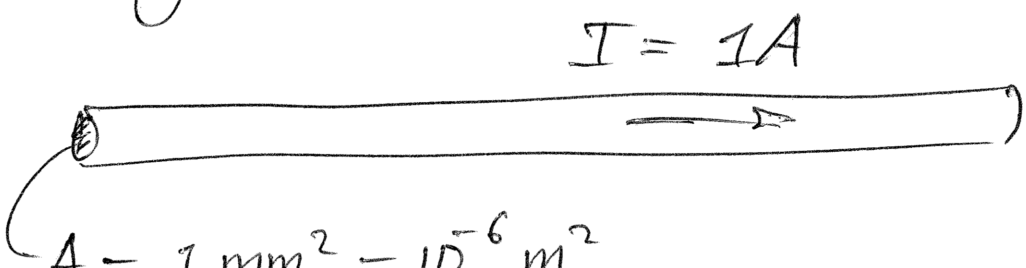
So :

$$\vec{\nabla} \cdot \vec{J} = 0 \quad \text{in steady-state}$$

In many conductors $n\, q\, \mu$ are constant throughout the material; when combined with the proportionality $\vec{J} = n q \mu \vec{E}$, we see that $\vec{E}$ is <u>divergence-less</u> in conductors, just as $\vec{J}$. This means that in a steady flow state the electric potential in a conductor satisfies the Laplace equation $\nabla^2 \varphi = 0$.

We now turn to the friction mechanism that is responsible for the drift velocity being proportional to the magnitude of the electric field. The first microscopic theory that applies to conduction in metals was worked out by Drude around 1900, decades before physicist knew about quantum mechanics. Being a classical theory (of highly "quantum" entities) has limitations, but it gets many things right. Our treatment will be very condensed and qualitative for the most part.

Let's see if we can even explain the order of magnitude of v_D. Here's a quick estimate of v_D when 1A of current is flowing in a copper wire:



$$v_D = \frac{I}{nqA} \qquad q = e \cong 10^{-19}\,C$$

$$n = \frac{\text{no. Cu atoms}}{\text{Vol}} \cong 10^{29}\,m^{-3}$$

(1 free (conduction) electron per Cu)

$$v_D \simeq \frac{1}{10^{29} \times 10^{-19} \times 10^{-6}} = 10^{-4}\ m/s$$

So in the short time (humanly imperceptible) between flipping a switch and observing a bulb light up, each electron has drifted only a very tiny fraction of the length of wire in the light-bulb circuit!

In 1900 physicist already knew how to calculate the random speeds of particles when they are free to exchange energy with a medium at some temperature T:

v_T = thermal speed of an electron at temp. T

$$\left\langle \tfrac{1}{2} m_e v_T^2 \right\rangle = \tfrac{3}{2} k_B T$$

average

(Law of equipartition of thermal energy)

k_B = Boltzmann's constant
$\simeq 10^{-23}$ J/Kelvin

Find v_T for $T = 300$ Kelvin :

$$v_T \simeq \sqrt{\frac{k_B T}{m_e}} \simeq \sqrt{\frac{10^{-23} \times 300}{10^{-30}}}$$

$$\simeq 10^5 \text{ m/s} \; !$$