Lecture 11

To prove the uniqueness of the solutions to Laplace's equation

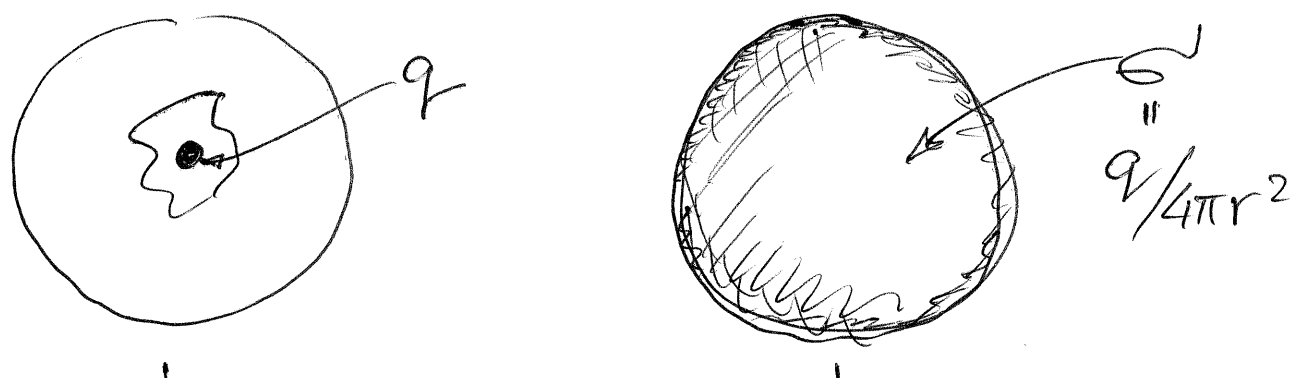
$$\nabla^2 \varphi = 0$$

given the values of φ on some boundary surfaces (conductors) we need to use the following fact:

[The average of φ over the surface of a sphere is equal to its value at the sphere center.]

This is similar in character to what we found for the values of φ on a grid, when the Laplace equation is solved numerically. But this sphere property applies to any sphere, not just the limit of small spheres, as long as the sphere is in a region with $\rho(\vec{r})=0$.

A physical restatement of this spherical-average property is the following. Consider a sphere $\mathcal{S}$ in a region of space where $\rho=0$ but where there may be a static electric field, which we describe in terms of its ~~electrost~~ potential Φ. Suppose we move a charge q from some reference point (e.g. infinity) to the center of $\mathcal{S}$. Alternatively, we can move that same quantity of charge to the spherical surface $\mathcal{S}$ in the form of a uniform surface charge density :



$$U_1 = q\,\varphi(0) \qquad\qquad U_2 = q\,\langle \varphi \rangle_{surf.}$$

$$\langle \varphi \rangle_{surf} = \left(\begin{array}{c} \text{average value of } \varphi \\ \text{on the spherical surface} \end{array} \right)$$

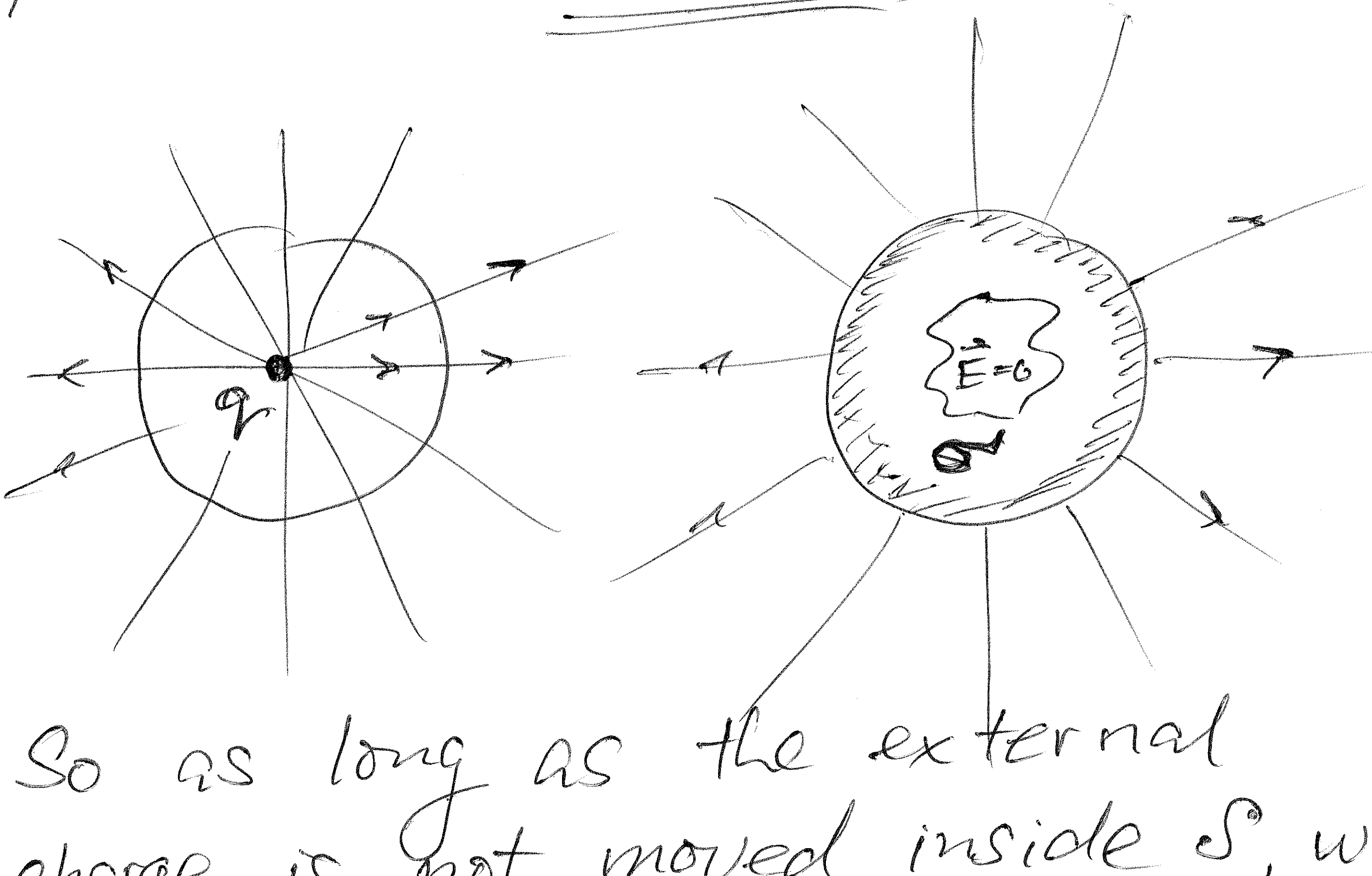
If U_1 is always equal to U_2, then the average of φ over the surface is always equal to its value at the center.

To see why U_1 is always equal to U_2, we shift our focus to the charges responsible for

the electric field and φ.
These could be charges on the surfaces of conductors or, more generally, any charges outside the region we are investigating. Consider just one of those "external" charges; by assumption it would be located outside the sphere S. The external charge contributes to the energy U_1 and U_2 through the electric field it produces, as do all the other external charges. We can work out the contribution to the energy for one external charge by bringing it from infinity and calculating the work required to move it to its actual position (on the surface of a conductor, etc.).

The work relevant to our charge q, either in the form of a point or spread out uniformly over the sphere surrounding this point, ~~is~~ results from the electric field produced by q.

But outside S, both of these produce the same $\vec{E}$!



So as long as the external charge is not moved inside S, we get exactly the same work.

We now turn to the question of uniqueness of solutions of the Laplace equation. Consider the problem we solved numerically in last lecture. The value of φ was specified on both the top and bottom boundaries. Suppose the algorithm had found two solutions (say for different initial values), then

$$\nabla^2 \varphi_1 = 0$$

$$\nabla^2 \varphi_2 = 0$$

and $\varphi_1 = \varphi_2$ on all the boundaries. Subtracting the equations,

$$\nabla^2 (\underbrace{\varphi_1 - \varphi_2}) = 0$$

$$\Rightarrow \nabla^2 \varphi_3 = 0$$

where φ_3 is a third solution, but one which has $\varphi_3 = 0$ on all the boundaries (since $\varphi_3 = \varphi_1 - \varphi_2 = 0$ there). But this would mean that φ_3, a solution of Laplace's equation, has a <u>maximum</u> or <u>minimum</u> somewhere in the space between the boundaries (it can't be identically zero since then $\varphi_1 = \varphi_2$ means we did not have two different solutions). This is <u>impossible</u> by the spherical-average property: place the center of the sphere at the claimed maximum or minimum; the average of φ_3 ~~on~~ over the sphere surface will then not equal φ_3 at the center.