Lecture 10

Let's examine the differential form of Gauss's law when $\vec{E}$ is expressed in terms of the electric potential:

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$

$$\downarrow$$

$$\vec{\nabla} \cdot (-\vec{\nabla} \varphi) = \rho/\epsilon_0$$

The divergence of a gradient is called the "Laplacian":

$$\vec{\nabla} \varphi = \frac{\partial \varphi}{\partial x} \hat{x} + \frac{\partial \varphi}{\partial y} \hat{y} + \frac{\partial \varphi}{\partial z} \hat{z}$$

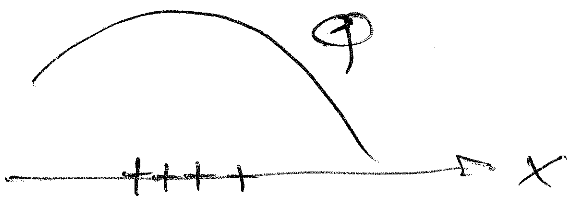
$$\vec{\nabla} \cdot (\vec{\nabla} \varphi) = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}$$

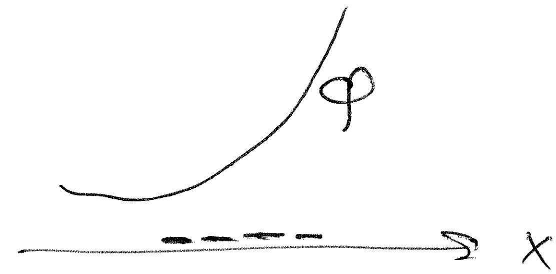
$\vec{\nabla} \cdot (\vec{\nabla} \varphi)$ is abbreviated as $\nabla^2 \varphi$. Gauss's law, written in terms of φ, is then

$$\nabla^2 \varphi = -\rho/\epsilon_0$$

Roughly speaking, the potential "curves" negatively or positively, depending on the sign of the charge density. Suppose ρ only varies with x and we can assume the same for φ; then

$$\frac{\partial^2 \varphi}{\partial x^2} = -\rho/\epsilon_0$$





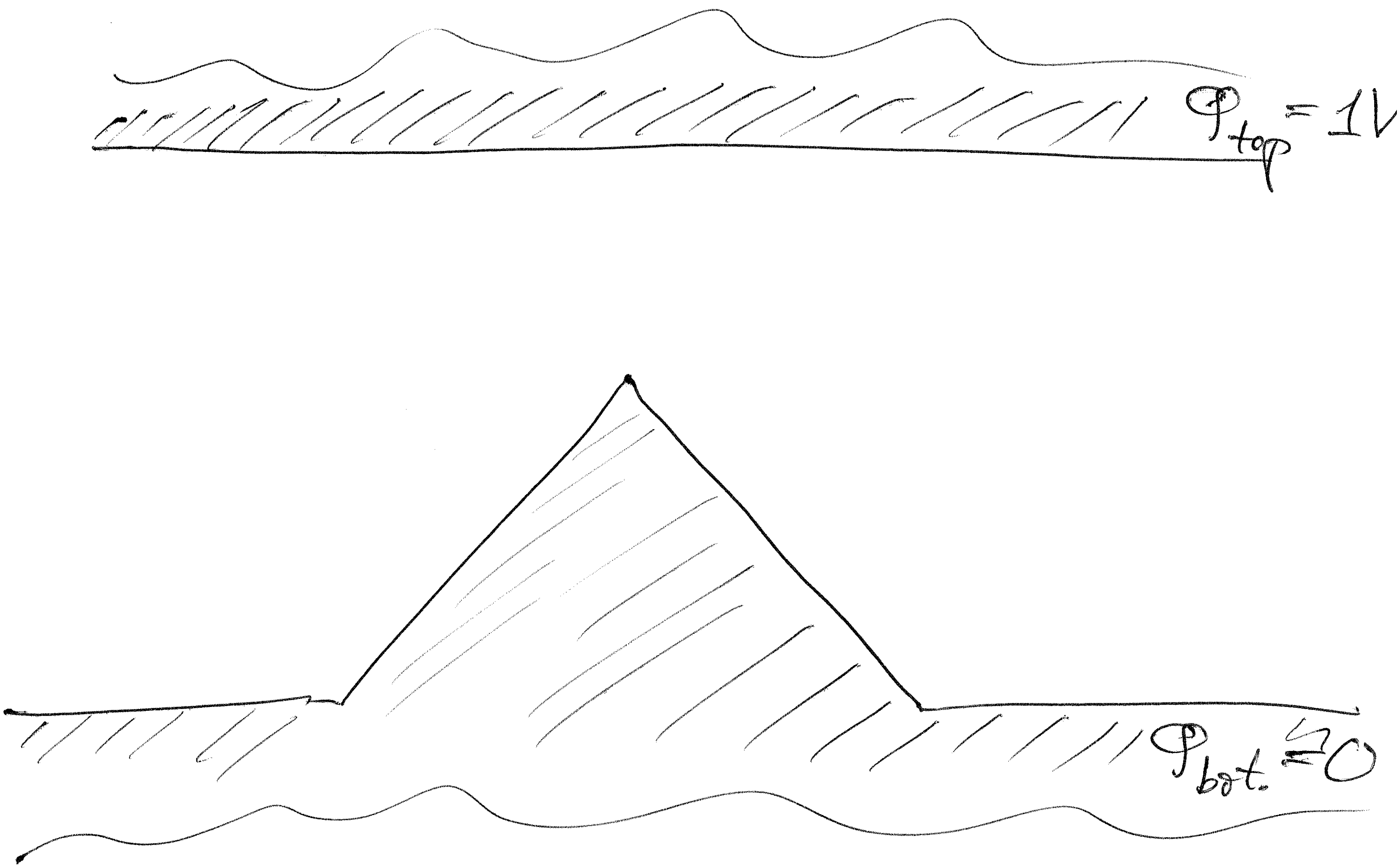
When $\rho = 0$, φ doesn't "curve" at all. What exactly does this mean? This is an important special case, one that deserves careful study. It applies, for example, to situations where we have some conductors in a space that is free of charges. Outside the conductors we have $\rho = 0$ and therefore

$$\nabla^2 \varphi = 0 \quad \text{"Laplace equation"}$$

The value of φ on each conductor is a constant, in general a different value on each conductor. And since φ is a <u>continuous</u> function — otherwise $\vec{\nabla}\varphi = -\vec{E}$ would be divergent/unphysical — the equation $\nabla^2\varphi = 0$ must determine how the potential varies in space so as to match the values on the conductors. A concrete example is useful at this point. Our example is two-dimensional to keep the drawings simple; it also bears on the physics of lightning rods.

Consider the space between

a flat top conductor and a bottom conductor with a sharp wedge:



Further suppose the potentials φ_{top} and $\varphi_{bot.}$ are set at different values, say differing by $\Delta\varphi = 1$ Volt.

We would like to determine φ between the two conductors and map out the contours for potential between 0 and 1 Volt. All we know at this point is that near the conductors these contours will be parallel to the conductor surface.

We can solve this problem numerically by setting up the Laplace equation on a grid:

$$
\begin{array}{ccccccc}
\cdots & — & \varphi(x,y+b) & — & \cdots \\
| & & | & & | \\
\varphi(x-b,y) & — & \varphi(x,y) & — & \varphi(x+b,y) \\
| & & | & & | \\
\cdots & — & \varphi(x,y-b) & — & \cdots
\end{array}
$$

As you know, the finite difference approximation to the second derivative of a function (with respect to x) takes the form

$$\frac{\partial^2 \varphi}{\partial x^2} \cong \frac{\varphi(x+b,y) - 2\varphi(x,y) + \varphi(x-b,y)}{b^2}$$

Adding the analogous expression for $\frac{\partial^2 \varphi}{\partial y^2}$ to get the Laplacian, the Laplace equation simplifies to the following:

$$\nabla^2 \varphi = 0$$

$$\Downarrow$$

$$\varphi(x+b,y) + \varphi(x-b,y) + \varphi(x,y+b) + \varphi(x,y-b)$$

$$= 4\varphi(x,y)$$

So on the grid, a solution is characterized by the property that the value of φ at (x,y) is just the average of the values at the four neighboring grid points, ($(x+b,y)$, etc.)! This property is the basis of a surprisingly simple algorithm for determining φ:

(1) set φ to the conductor potentials at grid points within the conductors

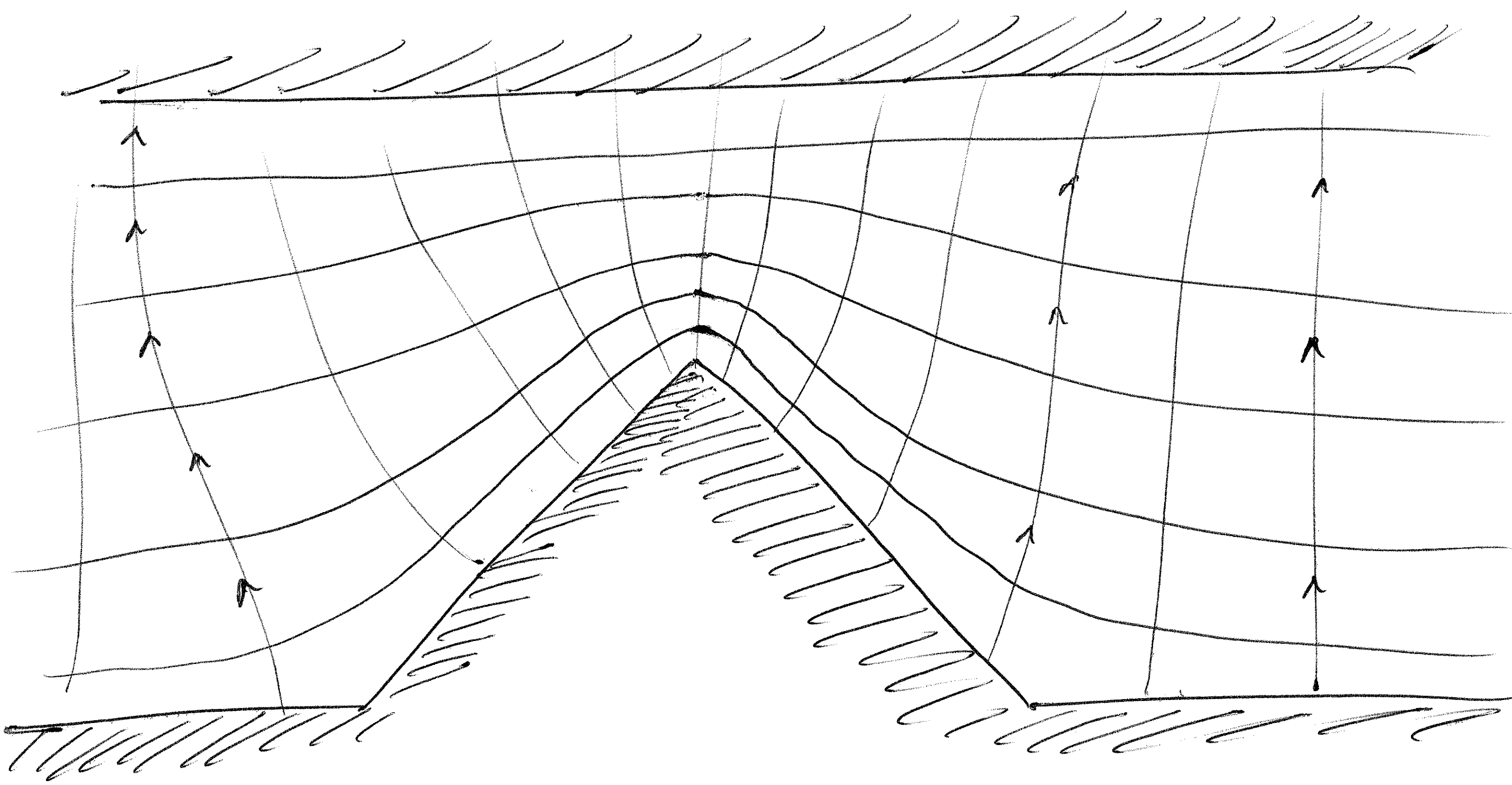
(2) set φ at grid points outside the conductors to some initial values (random values work, but may not be the best choice)

(3) replace φ at each grid point outside the conductors by the average of its neigbors; repeat until φ stops changing.

The termination criterion, "until φ stops changing", happens only when φ satisfies the grid approximation of the Laplace equation.

Even when this algorithm produces a solution (i.e. φ stops changing after some number of iterations of step 3) we might well ask if our solution is <u>unique</u> — might we have obtained a different solution had we started with different initial values? Fortunately the solutions of the Laplace equation <u>are</u> unique (given a set of boundary values on conductors or possibly "at infinity") so this numerical method will work.

We will prove uniqueness in the next lecture. For now, let's return to our 2D problem and examine our solution:



The φ = const. contours are more compressed around the wedge — they have no choice — and the electric field therefore has a

greater magnitude there.
More interestingly, the contours are especially tight at the tip of the wedge. This is a general property of conductors with sharp edges/corners: the electric field is exceptionally strong at these sharp features.
The functionality of lightning rods is based on this fact, though not in the way most people expect (subject of future lecture).