Extended summary of the renormalization group calculation of percolation exponents

Resizing and rescaling after tracing-out modes in the momentum shell

After tracing-out modes in the momentum shell the field has the Fourier representation

$$\Psi_i^<(x) = \sum_{p\in\Lambda^*}^{<} e^{ip\cdot x}\ \hat{\Psi}_i(p),$$

where the $<$ over the summation means up to modes with the smaller cutoff, K/b. Next, rewrite the sum as a sum over modes in the expanded dual lattice, with cutoff K:

$$\Psi_i^<(x) = \sum_{p'\in b\Lambda^*} e^{i(p'/b)\cdot x}\ \hat{\Psi}_i(p'/b).$$

To restore the sum to a sum over the original dual lattice (now with cutoff K) we do two things. (1) Introduce "interpolating" Fourier coefficients at modes $p \in \Lambda^*$

$$\hat{\Psi}'_i(p)$$

with matching values where the dual lattices coincide:

$$p \in b\Lambda^* : \quad \hat{\Psi}'_i(p) = \hat{\Psi}_i(p/b).$$

(2) We must also compensate the higher density of modes (in the sum over Λ^*) with the factor b^{-D}. The result is

$$\begin{aligned}\Psi_i^<(x) &= \frac{1}{b^D}\sum_{p\in\Lambda^*} e^{i(p/b)\cdot x}\ \hat{\Psi}'_i(p)\\ &= \frac{1}{b^D}\Psi'_i(x/b).\end{aligned}$$

Finally, we introduce a rescaling factor χ that will be used to restore the coefficient $c = 1$ for the gradient term:

$$\Psi_i^<(x) = \frac{\chi}{b^D}\Psi'_i(x/b). \tag{1}$$

When the fields Ψ'_i appear integrated in the Hamiltonian we make the change of variables $x' = x/b$ and use

$$d^D x = b^D\ d^D x'$$

The same change of variables applied to the squared gradient gives

$$\nabla\cdot\nabla = \frac{1}{b^2}\ \nabla'\cdot\nabla'.$$

Resizing and rescaling applied to the Potts model

After tracing out modes in the momentum shell, the Potts model Hamiltonian has the form

$$H^{<} = \int d^D x \left(\frac{c^{<}}{2}\, \nabla\Psi_i^{<} \cdot \nabla\Psi_i^{<} + \frac{r^{<}}{2}\, \Psi_i^{<}\Psi_i^{<} - w^{<}\, Q_{ijk}\, \Psi_i^{<}\Psi_j^{<}\Psi_k^{<} - h\, \mathbf{e}_i^1 \Psi_i^{<} \right).$$

The magnetic field parameter h is unchanged ($h^{<} = h$) by tracing-out because it couples only to the zero momentum mode of the field.

Defining H' by the resizing and rescaling transformations applied to $H^{<}$, we obtain the following transformation rules for the parameters

$$\begin{aligned} r' &= \chi^2 \frac{r^{<}}{b^{(2-1)D}} \\ w' &= \chi^3 \frac{w^{<}}{b^{(3-1)D}} \\ h' &= \chi^1 h\ , \end{aligned}$$

where χ is determined by the condition $c' = 1$:

$$\chi = \sqrt{\frac{b^{D+2}}{c^{<}}}.$$

Parameter transformations for an infinitesimal scale change

Renormalization group flow is parameterized by the logarithmic scale factor $\lambda = \log b$. The infinitesimal change in scale $\delta\lambda$ generated by tracing out modes in the momentum shell of thickness

$$\delta K = K - K/b = K\,\delta\lambda$$

changes the parameter as follows:

$$\begin{aligned} c^{<} &= 1 + \left(\frac{\delta c}{\delta\lambda}\right)\delta\lambda \\ r^{<} &= r(\lambda) + \left(\frac{\delta r}{\delta\lambda}\right)\delta\lambda \\ w^{<} &= w(\lambda) + \left(\frac{\delta w}{\delta\lambda}\right)\delta\lambda \\ h^{<} &= h(\lambda). \end{aligned}$$

After resizing and rescaling, the resulting parameters define

$$\begin{aligned} r(\lambda+\delta\lambda) &= r' \\ w(\lambda+\delta\lambda) &= w' \\ h(\lambda+\delta\lambda) &= h'\ . \end{aligned}$$

Expanding r', w' and h' to linear order in $\delta\lambda$ one obtains the flow equations

$$\begin{aligned}
\dot{r} &= 2r - r\left(\frac{\delta c}{\delta\lambda}\right) + \left(\frac{\delta r}{\delta\lambda}\right) && (2)\\
\dot{w} &= (3 - D/2)w - \frac{3}{2}w\left(\frac{\delta c}{\delta\lambda}\right) + \left(\frac{\delta w}{\delta\lambda}\right) && (3)\\
\dot{h} &= \frac{1}{2}\left(D + 2 - \left(\frac{\delta c}{\delta\lambda}\right)\right) h\,, && (4)
\end{aligned}$$

where $\dot{r} = dr/d\lambda$, etc.

For some things one needs the flow of the fields as well. Using (1) with $b = 1 + \delta\lambda$ and replacing the superscripts $<$ and $'$ with λ and $\lambda + \delta\lambda$ we obtain

$$\Psi_i^{\lambda+\delta\lambda}(x/b) = \left(1 + \frac{\delta\lambda}{2}\left(D - 2 + \left(\frac{\delta c}{\delta\lambda}\right)\right)\right)\Psi_i^{\lambda}(x). \qquad (5)$$

Results of the cubic-term perturbation calculation

The tracing over modes in the momentum shell, performed via diagramatic perturbation theory applied to the cubic term of the Potts Hamiltonian, results in the following three inputs to the flow equations:

$$\begin{aligned}
\frac{\delta c}{\delta\lambda} &= \frac{36}{D}(q-2)w^2\Omega_D\frac{K^{D+2}}{(K^2+r)^4}\\
\frac{\delta r}{\delta\lambda} &= -18(q-2)w^2\Omega_D\frac{K^{D}}{(K^2+r)^2}\\
\frac{\delta w}{\delta\lambda} &= 36(q-3)w^3\Omega_D\frac{K^{D}}{(K^2+r)^3}\,.
\end{aligned}$$

Rationalized parameters, percolation limit, and $D = 6 - \epsilon$

In analyzing the flow equations (2,3,4) we start with the case where at the original scale, and therefore also at all larger scales, $h = 0$. This will be revisited later, when we analyze the free energy. To tidy up the equations for the flow in the (r, w) plane, we absorb some constant factors by defining "rationalized" parameters:

$$\begin{aligned}
\tilde{r} &= r/K^2\\
\widetilde{w}^2 &= \Omega_D K^{D-6} w^2.
\end{aligned}$$

Along with these definitions, we make the following simplifications that apply specifically to the percolation limit of the Potts model:

1. Set $q = 1$ and $D = 6-\epsilon$. The deviation ϵ from the upper critical dimension is treated as a small parameter and we discard higher order terms.

2. The factors $(K^2 + r)^{-n}$ are expanded for $r \ll K^2$, since we are only interested in the flow in the vicinity of the fixed point where $r = O(\epsilon)$.

Here are the resulting flow equations (accents on the rationalized parameters have been dropped):

$$\begin{aligned}\dot{r} &= 2r + 18w^2 - 30rw^2 + O(r^2w^2, w^4) \\ \dot{w} &= \frac{\epsilon}{2}w - 63w^3 + O(rw^3, w^5)\ .\end{aligned}$$

The omitted terms are higher order in r/K^2, or terms that arise from higher order perturbation diagrams (four or more vertices).

Fixed points and the critical state

For $\epsilon < 0$ the flow equations have only $(r^*, w^*) = (0, 0)$ as a fixed point, called the Gaussian fixed point. There is an additional fixed point when $\epsilon > 0$, the Wilson-Fisher fixed point, located at

$$\begin{aligned}r^* &= -\frac{\epsilon}{14} \\ w^* &= \sqrt{\frac{\epsilon}{126}},\end{aligned}$$

to leading order in small ϵ.

In either case, when the model parameters are set at (r^*, w^*), the system is scale invariant because it is unchanged by the renormalization group transformation. This critical state is characterized by the exponent

$$\begin{aligned}\eta &= \left.\frac{\delta c}{\delta\lambda}\right|_* \\ &= -\left.6w^2\right|_*\end{aligned} \tag{6}$$

where the second line is expressed in terms of rationalized parameters as before. The Gaussian critical state has $\eta = 0$, while the Wilson-Fisher critical state has

$$\eta = -\frac{\epsilon}{21} + O(\epsilon^2)\ .$$

Composing the infinitesimal field rescalings in (5), when $\delta c/\delta\lambda$ takes the constant fixed point value, results in

$$\begin{aligned}\Psi_i^{\lambda+\lambda_0}(x/b) &= \exp\left(\frac{\lambda}{2}(D-2+\eta)\right)\ \Psi_i^{\lambda_0}(x) \\ &= b^{(D-2+\eta)/2}\ \Psi_i^{\lambda_0}(x)\ .\end{aligned} \tag{7}$$

We can use this to find field correlations in the critical state. For example, consider

$$C^*_{ij}(x) = \langle\Psi_i^0(0)\ \Psi_j^0(x)\rangle\ ,$$

the two-point correlation function of the original system ($\lambda = 0$) with critical parameters. Using (7) rewrite this in terms of the fields of the critical model at scale $b = e^{\lambda}$, and define $x_0 = x/b$:

$$\begin{aligned} C^*_{ij}(x) &= b^{-D+2-\eta} \, \langle \Psi^{\lambda}_i(0) \; \Psi^{\lambda}_j(x/b) \rangle \\ &= \left(\frac{\|x_0\|}{\|x\|} \right)^{D-2+\eta} \langle \Psi^{\lambda}_i(0) \; \Psi^{\lambda}_j(x_0) \rangle . \end{aligned}$$

We fix x_0 at some small distance, say one lattice spacing, so that the expectation value can be treated as a constant. The large $\|x\|$ behavior of the correlation function at criticality,

$$C^*_{ij}(x) \propto \frac{1}{\|x\|^{D-2+\eta}} ,$$

establishes the RG exponent η as the correction relative to the pure squared-gradient model. Applied to percolation, below six dimensions, we see that the epsilon correction produces a slower ($\eta < 0$) decay of correlations in the critical cluster.

Flow near the fixed point and the approach to criticality

The flow equations, when linearized about the Wilson-Fisher fixed point, are represented by the following matrix, up to terms of order ϵ:

$$\begin{bmatrix} \frac{\partial \dot{r}}{\partial r}\Big|_* & \frac{\partial \dot{r}}{\partial w}\Big|_* \\ \frac{\partial \dot{w}}{\partial r}\Big|_* & \frac{\partial \dot{w}}{\partial w}\Big|_* \end{bmatrix} = \begin{bmatrix} 2 - \frac{5}{21}\epsilon & 36\sqrt{\frac{\epsilon}{126}} \\ 0 & -\epsilon \end{bmatrix}$$

The positive eigenvalue, quantifying the flow toward large r, of either sign, defines the reciprocal of the correlation length exponent:

$$\nu = \left(2 - \frac{5}{21}\epsilon \right)^{-1} = \frac{1}{2} + \frac{5}{84}\epsilon + O(\epsilon^2) .$$

Defining $\delta r'$ as the coordinate along the unstable axis (with origin at the fixed point), we obtain

$$\delta r'(\lambda_0 + \lambda) = \delta r'(\lambda_0) \exp\left(\lambda/\nu\right) . \tag{8}$$

Even though the flow eventually leaves the region where the linearization is a good approximation, if $\delta r'(\lambda_0)$ is tuned very close to zero, then most of the incremental rescaling (on the logarithmic λ scale) occurs in the fixed point neighborhood where linearization holds.

Controlling criticality and the divergence of lengths

At the scale of the original Potts model, the w parameter is not small and the system is far from the Wilson-Fisher fixed point where $w^* = O(\sqrt{\epsilon})$. On the other hand, a fixed (and modest) level of rescaling will flow the original parameters $(r_0 + \delta r, w_0)$ into a small enough neighborhood of the fixed point such that the linearized flow equations are a good approximation. The value $\delta r = 0$ identifies the separatrix in the flow. For $\delta r > 0$ the flow is toward large positive r, or a large scale model where Ψ has a vanishing mean value, while the large scale model favors a non-vanishing Ψ when $\delta r < 0$. Under RG flow by some fixed scale b_0 the value of δr maps to the coordinate $\delta r'$ along the unstable direction of the fixed point. Linearizing this regular map, so $\delta r' = A\delta r$, and using (8) we obtain the equation

$$\delta r'(\lambda) = b^{1/\nu} A\delta r, \tag{9}$$

after additional scaling by the factor $b = e^{\lambda}$. However closely we tune the transition with δr, this equation tells us what amount of rescaling b is required to flow $\delta r'$ some fixed distance $d = |\delta r'(\lambda)|$ from the fixed point. Keeping in mind that δr can have either sign, we obtain

$$b = B|\delta r|^{-\nu}, \tag{10}$$

where B combines the constants A and d. Models that flow to $\delta r'(\lambda) = \pm d$ are essentially unique (for either sign) because the w parameter is attracted to the stable axis of the fixed point. These models therefore have identical characteristics (again, for either sign). If the feature is a length, such as the decay length of the (non-critical) correlation function, we must remember that this length should be multiplied by the rescaling factor b if it is to describe the same feature in the original model. Equation (10) therefore gives the power law for the divergence of lengths in the model.

Free energy and the percolation exponents

Our handle on the percolation exponents is through the dependence of the free energy on the parameters δr and h. The signs of these control the transition in the mean value of the field Ψ. We therefore write

$$F = F_0 + F_1(\delta r, h),$$

where the nonsingular contribution F_0 may be taken as a constant in the domain of interest where both δr and h are small. Keeping with the spirit of RG, we don't calculate F_1 but relate it to the F_1 of a model at the larger scale $b = e^{\lambda}$. We already learned that (9) should be used for the δr parameter that applies at scale b. Using (4) and the definition (6) of η, the corresponding rescaling of h is

$$h(\lambda) = b^{(D+2-\eta)/2} h(0)$$

The relation between the F_1's at the two scales is therefore

$$F_1(\delta r, h) = \frac{1}{b^D} F_1(b^{1/\nu}\ \delta r, b^{(D+2-\eta)/2}\ h)\ ,$$

where the factor b^D corrects for the volume increase of the model at scale b.

To obtain the magnetization and susceptibility we take respectively one and two derivatives with respect to h, set $h = 0$, and as in the discussion of diverging lengths, select b such that

$$b^{1/\nu}\ \delta r = \pm\delta r_0$$

for some fixed $\delta r_0 > 0$. The result of those calculations is

$$\begin{aligned} m &= \left(\frac{|\delta r|}{\delta r_0}\right)^{\nu(D-2+\eta)/2} \partial_h F_1(\pm\delta r_0, 0) \\ \chi &= \left(\frac{\delta r_0}{|\delta r|}\right)^{\nu(2-\eta)} \partial_h^2 F_1(\pm\delta r_0, 0)\ , \end{aligned}$$

and identifies the following combination of RG exponents as the critical exponents of percolation:

$$\begin{aligned} \beta &= \nu\,(D-2+\eta)/2, \\ \gamma &= \nu\,(2-\eta). \end{aligned}$$

Finally, substituting $D = 6 - \epsilon$ and the result of the small epsilon calculation of ν and η,

$$\begin{aligned} \beta &= 1 - \frac{\epsilon}{7} + O(\epsilon^2), \\ \gamma &= 1 + \frac{\epsilon}{7} + O(\epsilon^2). \end{aligned}$$