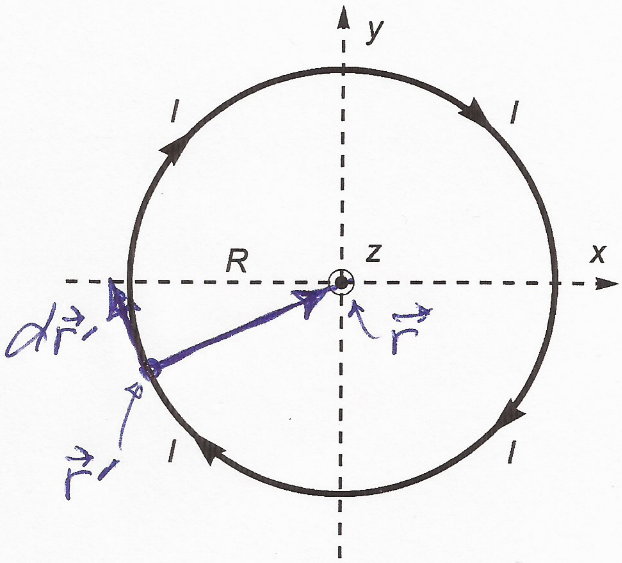


1. *(15 points)* Current I flows in a circular coil C of radius R as shown above.

(a) Using the Biot-Savart formula,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int_C \frac{d\mathbf{r}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

determine the **direction** of B at the center of the coil. Explain your reasoning.

$d\vec{r}' \times (\vec{r} - \vec{r}') = |d\vec{r}'| R (-\hat{z})$

into the page, $-\hat{z}$

(b) Using the same formula, determine the **magnitude** B of the magnetic field at the center of the coil. The steps in your calculation should be easy to follow. Express B in terms of I, R, and μ_0.

$$d\vec{r}' \times (\vec{r} - \vec{r}') = -|d\vec{r}'| R \, \hat{z}$$

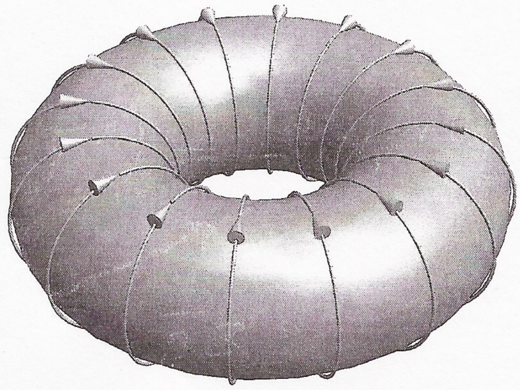
$$\frac{1}{|\vec{r} - \vec{r}'|^3} = \frac{1}{R^3}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint_C \left(- \frac{|d\vec{r}'|}{R^2} \right) \hat{z}$$

$$= - \frac{\mu_0 I}{4\pi R^2} \hat{z} \underbrace{\oint_C |d\vec{r}'|}_{2\pi R}$$

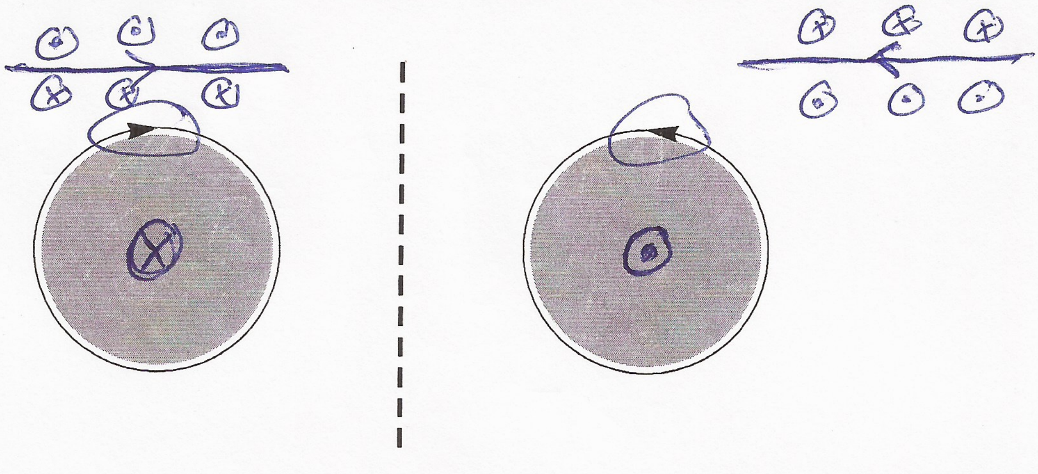
$$= - \frac{\mu_0 I}{2R} \hat{z}$$

$$B(\cancel{\vec{r}}0) = \frac{\mu_0 I}{2R}$$



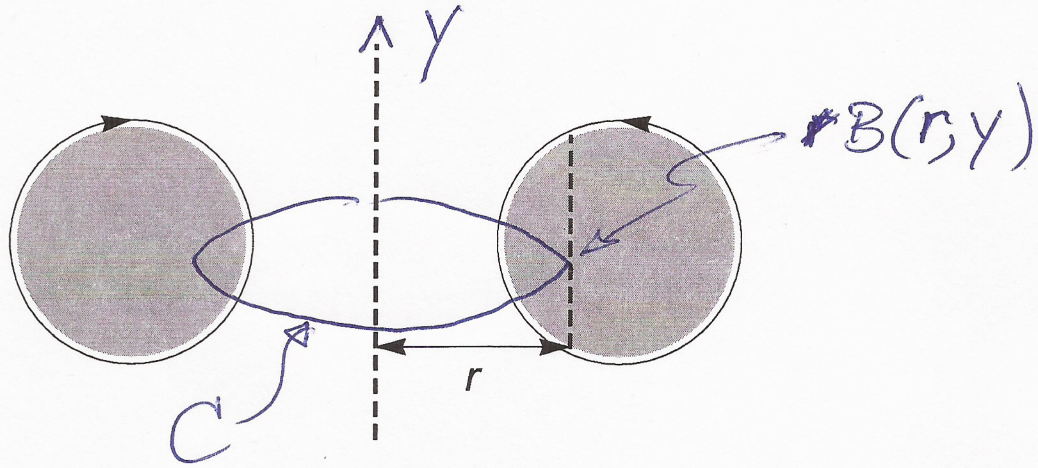
2. *(15 points)* Suppose there is a uniform current density flowing over the surface of a bagel as shown above, and that **the net current flowing through the hole of the bagel is** I. Here is a vertical cut through the bagel, also showing the axis of the bagel:

use straight wire comparison



One can show (you are not asked to do this) that the magnetic field $\mathbf{B}$ generated by this particular surface current density is purely azimuthal and non-zero only in the interior of the bagel. In other words, $\mathbf{B}$ can only point into or out-of the page and is nonzero only in the gray regions.

(a) Indicate with a $\otimes$ (into) or $\odot$ (out-of) the direction of $\mathbf{B}$ in the two regions above.



(b) Use the integral form of Ampère's law to argue that the magnitude B of the magnetic field is constant along the dashed line on the right that has a fixed distance r from the axis. Be sure to specify the curve C in your application of Ampère's law.

C: circle whose axis is y-axis (dashed line) with radius r and height y

$$\oint_C \vec{B} \cdot d\vec{r} = B(r,y) \cdot 2\pi r$$

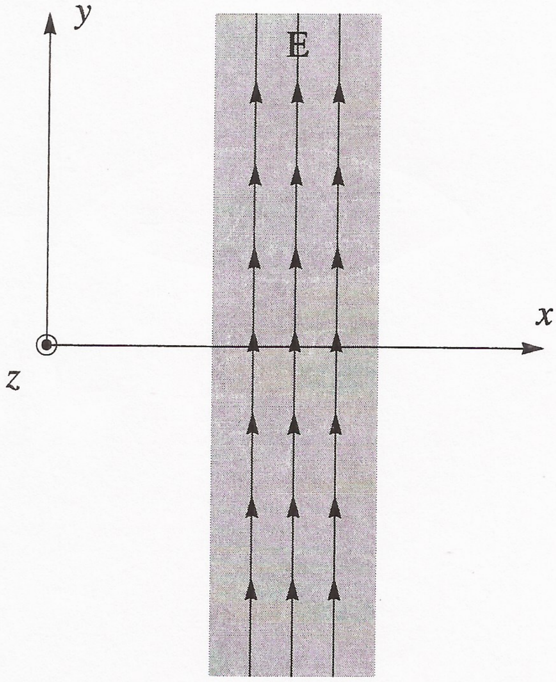
$$= \mu_0 I_{enc} = \mu_0 I$$

All C enclose the same current, so B(r,y) does not depend on y.

(c) Use the integral form of Ampère's law to calculate the field magnitude $B(r)$. You should be able to use the same kind of curve C you used in part (b). Express your answer in terms of the net current I, r and μ_0.

$$B(r) \cdot 2\pi r = \mu_0 I$$

$$B(r) = \frac{\mu_0 I}{2\pi r}$$



3. *(15 points)* As shown above, there is a uniform electric field $\mathbf{E} = E\hat{\mathbf{y}}$ within an infinite slab oriented so it is perpendicular to the x-axis. The electric and magnetic fields are zero outside the slab.

(a) What should be the direction of the uniform magnetic field inside the slab so the whole field configuration (**E** and **B**) moves to the **right** with velocity $c\hat{\mathbf{x}}$?

$\vec{S}$ should be in $+\hat{x}$ direction

Since $\vec{S} \propto \vec{E} \times \vec{B}$, $\vec{B}$ is in $+\hat{z}$ directio

(b) Relate the magnitude of **B** to E.

$|\vec{B}| = E/c$

Recall from HW 12, or use units

(c is only velocity given)

(c) Calculate the Poynting vector $\mathbf{S} = (1/\mu_0)\,\mathbf{E} \times \mathbf{B}$ inside the slab.

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\frac{E^2}{c}\hat{x} = c\epsilon_0 E^2\hat{x}$$

(d) The slab is incident on a detector of area A (in the y-z plane) that absorbs its energy. Express the energy absorbed U in terms of E, A, the width of the slab w, and fundamental constants.

method 1 : $U = |\vec{S}|AT$, $T = \frac{w}{c}$

$$= c\epsilon_0 E^2 A\frac{w}{c} = \epsilon_0 E^2 Aw$$

method 2 : $U = u \cdot \text{Vol.}$

$\frac{1}{\mu_0} = c^2\epsilon_0$

$$= \left(\frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right)Aw$$

$$= \frac{\epsilon_0}{2}\left(E^2 + c^2\left(\frac{E}{c}\right)^2\right)Aw$$

$$= \epsilon_0 E^2 Aw$$

(e) Using the general transformation rules for fields, in a primed frame moving with velocity $\mathbf{v} = v\,\hat{\mathbf{x}}$ relative to the unprimed frame,

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \qquad \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}$$
$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp}) \qquad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \tfrac{1}{c^2}\mathbf{v} \times \mathbf{E}_{\perp}),$$

calculate the fields $\mathbf{E}'$ and $\mathbf{B}'$ (all three components) in terms of E, c and v.

$(x = \parallel)$ $\quad E_x' = 0, \; B_x' = 0$

$$\vec{v} \times \vec{B}_{\perp} = (v\hat{x}) \times \left(\frac{E}{c}\hat{z}\right) = -\frac{v}{c}E\,\hat{y}$$

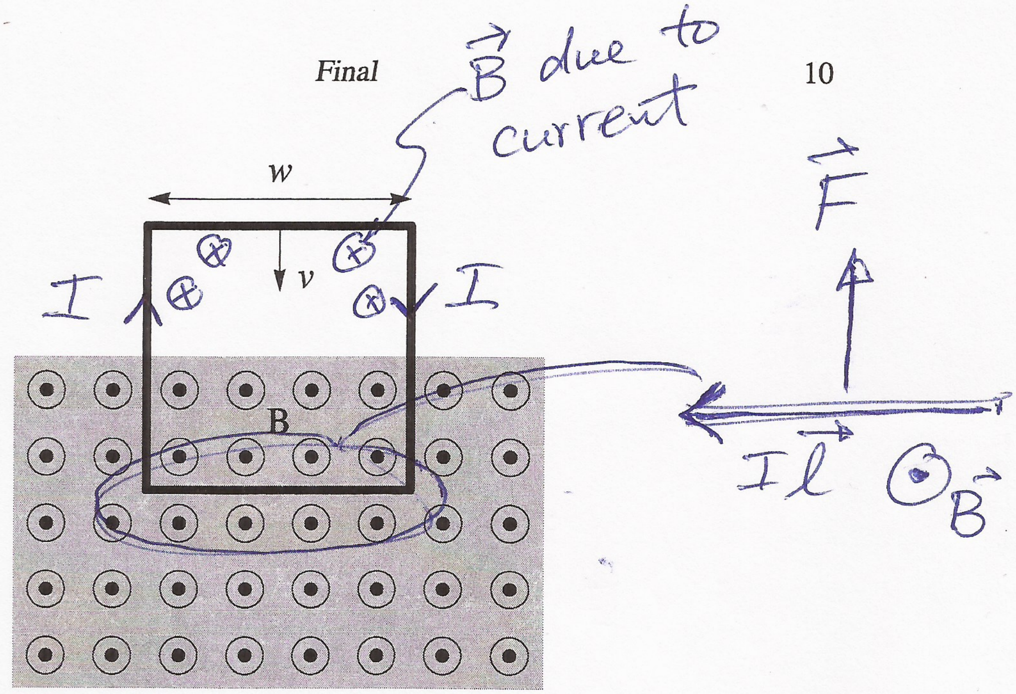
$$E_y' = \gamma\left(E - \frac{v}{c}E\right) = \gamma(1 - v/c)\,E$$

$$E_z' = 0$$

$$\vec{v} \times \vec{E}_{\perp} = (v\hat{x}) \times (E\hat{y}) = vE\,\hat{z}$$

$$B_z' = \gamma\left(\frac{E}{c} - \frac{1}{c^2}vE\right) = \gamma(1 - v/c)\,E/c$$

$$B_y' = 0$$



4. *(15 points)* The drawing above shows a square copper loop moving downward with speed v into the magnetic field of a strong magnet. Assume the field is zero outside the shaded region and uniform of magnitude B and out-of-the-page within the shaded region. The loop has side w and resistance R.

   (a) Determine the direction of current flow in the loop, clockwise or counterclockwise, using Lenz's law.

   Current produces $\vec{B}$ that counters the increasing (out-of-page) flux

   (b) Determine the magnitude of the current I using Faraday's law and Ohm's law.

$$\mathcal{E} = \frac{d\Phi_B}{dt} = Bwv$$

$$I = \mathcal{E}/R = \frac{Bwv}{R}$$

(c) On the diagram show the direction of the net magnetic force on the loop.

$\vec{F} = I\vec{\ell} \times \vec{B}$ upward

(d) Determine the magnitude of the net magnetic force F.

$$F = IwB = \frac{(Bw)^2 v}{R}$$

(e) Suppose the loop, of mass M, is actually falling in the gravitational field of the earth and has reached a constant terminal speed v_0 at the instant shown in the diagram. Determine v_0 in terms of M, g, R, w, and B.

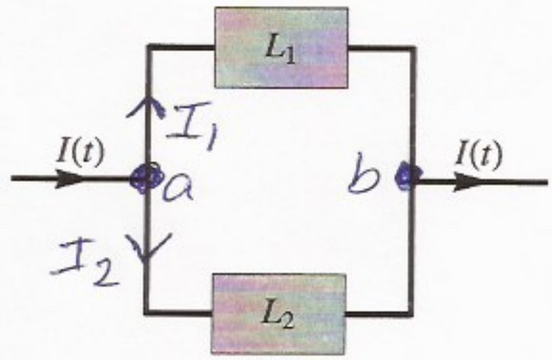
$$\frac{(Bw)^2 v_0}{R} = Mg$$ (zero acceleration, no net force)

$$v_0 = \frac{MgR}{(Bw)^2}$$

(f) Under the influence of gravity as in part (e), describe in words the nature of the loop's motion once the entire loop is completely within the region of uniform field.

$$\frac{d\Phi_B}{dt} = 0 \Rightarrow \mathcal{E} = 0 \Rightarrow I = 0 \Rightarrow \vec{F}_{mag} = 0$$

"freely falling"



5. *(10 points)* The diagram above shows two inductors L_1 and L_2 in a parallel arrangement. Starting from the voltage rule of inductors,

$$V_a - V_b = L\,\frac{dI}{dt}\,,$$

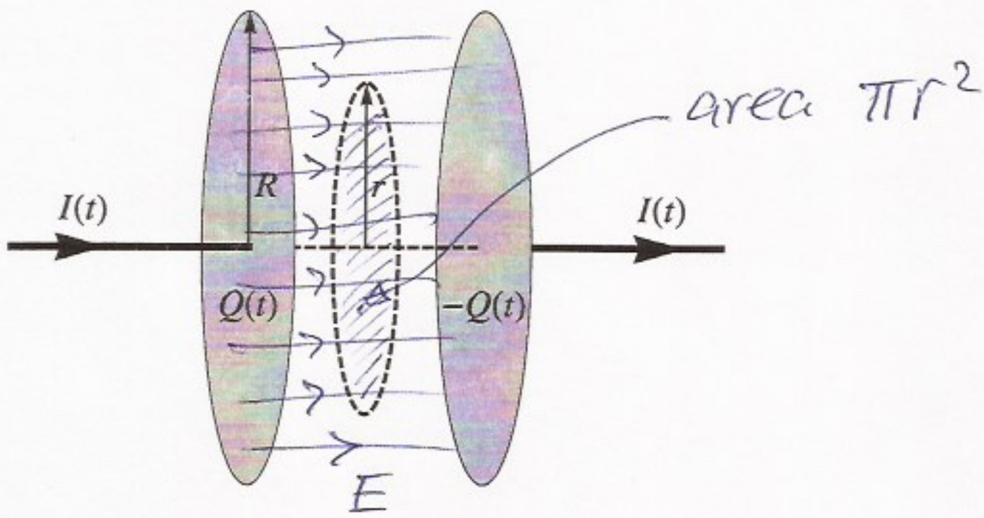
and Kirchhoff's circuit rules, show that the circuit above may be replaced with an equivalent lumped inductance L in the circuit below, and determine L in terms of L_1 and L_2.

$$V_a - V_b = \mathcal{E} = L_1 \dot{I}_1 = L_2 \dot{I}_2 = L \dot{I}$$

$$I_1 + I_2 = I \Rightarrow \dot{I}_1 + \dot{I}_2 = \dot{I}$$

$$\Rightarrow \frac{\mathcal{E}}{L_1} + \frac{\mathcal{E}}{L_2} = \frac{\mathcal{E}}{L}$$

$$\Rightarrow L = \left(\frac{1}{L_1} + \frac{1}{L_2}\right)^{-1}$$



6. *(15 points)* The drawing above shows current $I(t)$ flowing onto and off-of the plates of a capacitor. Edge effects may be ignored, because the radius R of the circular plates is much larger than the plate spacing (vertical scale is compressed in the drawing).

   (a) Calculate the flux of electric field $\Phi_E(r)$ through the surface $\mathcal{S}$ spanned by the circle $\mathcal{C}$ of radius r between the plates (shown dashed in the drawing) in terms of the charge $Q(t)$ on the left plate.

surface charge density on plate:

$$\sigma = Q/A = Q/\pi R^2$$

$$E = \sigma/\epsilon_0 = \frac{Q}{\epsilon_0 \pi R^2}$$

$$\Phi_E = E \cdot A_C \, , \quad A_C = \pi r^2$$

$$= \frac{Q(t)}{\epsilon_0}\left(\frac{r}{R}\right)^2$$

(b) Use the integral form of Ampère's law

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = \frac{1}{c^2} \int_S \frac{\partial \mathbf{E}}{dt} \cdot d\mathbf{a} + \mu_0 \int_S \mathbf{j} \cdot d\mathbf{a}$$

on the same curve/surface of part (a) to calculate the magnitude of magnetic field $B(r)$ between the plates at radius r from the center axis of the capacitor. You may assume the magnetic field is everywhere tangent to the circle. Express your answer in terms of r, R, I and μ_0.

(magnitudes) $B \cdot 2\pi r = \frac{1}{c^2} \frac{d\Phi_E}{dt}$

$$= \frac{1}{c^2} \frac{\dot{Q}}{\epsilon_0} \left(\frac{r}{R}\right)^2$$

$\dot{Q} = I$

$\frac{1}{c^2 \epsilon_0} = \mu_0$

$$B(r) = \frac{\mu_0 I}{2\pi r} \left(\frac{r}{R}\right)^2$$

(c) Compare the magnetic field you found in (b) between the plates with the magnetic field that encircles the wires outside the plates.

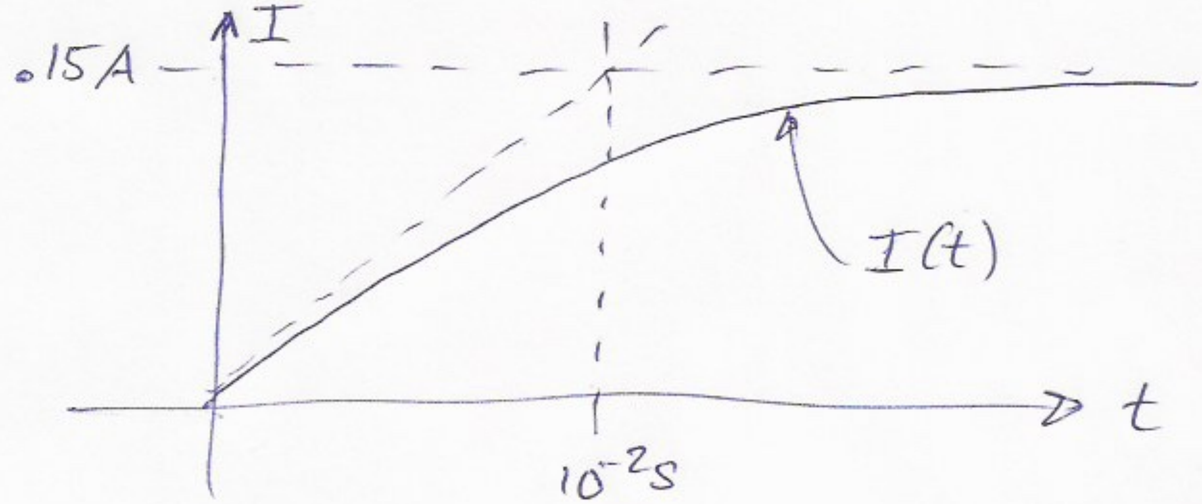
Same as B_{wire} apart from $(r/R)^2$ factor. $B(r) < B_{wire}$ for $r < R$

7. *(15 points)* A long coil of wire has resistance 10 Ω and inductance 0.1 Henry. At time $t = 0$ it is connected to a 1.5 V battery.

   (a) Plot the current $I(t)$ in the wire as a function of time, beginning at $t = 0$ and including the behavior at long times. Provide a scale on both axes (amps and seconds).

$$I(t) = \mathcal{E}/R\left(1 - e^{-t/\tau}\right)$$

$$\mathcal{E}/R = \frac{1.5}{10} = .15A, \quad \tau = L/R = \frac{.1}{10} = 10^{-2}\,\text{sec.}$$



   (b) Write an equation that relates the power provided by the battery, $P_{\mathcal{E}}$, the power consumed by the inductor (as magnetic energy) P_L, and the power dissipated in the resistor, P_R .

$$P_{\mathcal{E}} = P_L + P_R$$

   (c) Express the three power terms in (b) in terms of $\mathcal{E}$, $I(t)$, $\dot{I}(t)$ and R.

$$P_{\mathcal{E}} = \mathcal{E} I$$

$$P_L = (L\dot{I})I$$

$$P_R = (RI)I$$