Assignment 10 Solutions

$\vec{E}$ in moving frame

$$\vec{E} = E\hat{x} = \underbrace{\frac{E}{3}(\hat{x}+\hat{y}+\hat{z})}_{\vec{E}_{\parallel}} + \underbrace{\frac{E}{3}(2\hat{x}-\hat{y}-\hat{z})}_{\vec{E}_{\perp}}$$

(check: $\frac{1}{3}+\frac{2}{3}=1$, $\frac{1}{3}-\frac{1}{3}=0$, $\frac{1}{3}-\frac{1}{3}=0$)

$\vec{E}_{\parallel}$ is clearly parallel to $\vec{v} = \frac{c}{2}(\hat{x}+\hat{y}+\hat{z})$

$$\vec{E}_{\perp}\cdot\vec{v} = \left(\frac{E}{3}\right)\left(\frac{c}{2}\right)(2-1-1) = 0$$

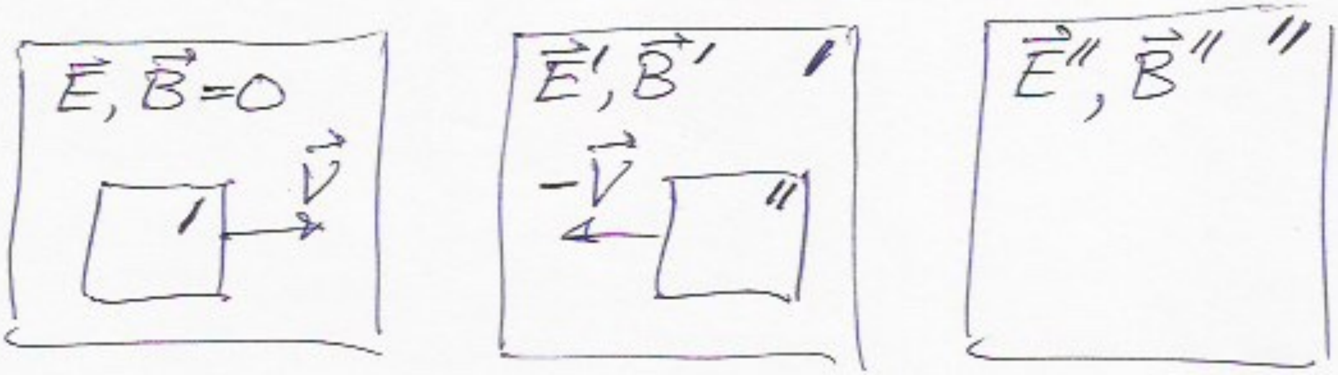
$$v^2 = \left(\frac{c}{2}\right)^2(1+1+1) = \frac{3}{4}c^2, \quad 1 - v^2/c^2 = \frac{1}{4}$$

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1/4}} = 2$$

$$\vec{E}' = \vec{E}_{\parallel} + \gamma\vec{E}_{\perp} = \frac{E}{3}(\hat{x}+\hat{y}+\hat{z}) + \frac{2}{3}E(2\hat{x}-\hat{y}-\hat{z})$$

$$= \frac{E}{3}(5\hat{x}-\hat{y}-\hat{z})$$

Compound Lorentz transformation



$$0 = \vec{E}_{\parallel} = \vec{E}_{\parallel}' = \vec{E}_{\parallel}'' \quad , \quad 0 = \vec{B}_{\parallel} = \vec{B}_{\parallel}' = \vec{B}_{\parallel}''$$

(a) $$\vec{E}_{\perp}' = \gamma(\vec{E}_{\perp} + \vec{V} \times \vec{0}) = \gamma \vec{E}_{\perp}$$

$$\vec{B}_{\perp}' = \gamma(\vec{0} - \tfrac{1}{c^2}\vec{V} \times \vec{E}_{\perp}) = -\tfrac{1}{c^2}\gamma \vec{V} \times \vec{E}_{\perp}$$

(b) We derived $\vec{E}_{\perp}' = \gamma \vec{E}_{\perp}$ by going into the rest frame of the moving charge (since we trusted the force law only for non-moving charges). Since the charge has velocity $\vec{V}' = 0$ in this frame, there is no additional force due to a magnetic field in this frame ($q\vec{V}' \times \vec{B}' = 0$).

(c) $\vec{E}_\perp'' = \gamma(\vec{E}_\perp' + (-\vec{v})\times\vec{B}_\perp')$

$$= \gamma^2\vec{E}_\perp + \frac{1}{c^2}\gamma^2\underbrace{\vec{v}\times(\vec{v}\times\vec{E}_\perp)}_{(\underbrace{\vec{v}\cdot\vec{E}_\perp}_{0})\vec{v} - (\underbrace{\vec{v}\cdot\vec{v}}_{v^2})\vec{E}_\perp}$$

$$\vec{E}_\perp'' = \gamma^2(1 - v^2/c^2)\vec{E}_\perp$$

$$= \vec{E}_\perp$$

$$\vec{B}_\perp'' = \gamma(\vec{B}_\perp' - \frac{1}{c^2}(-\vec{v}\times\vec{E}_\perp'))$$

$$= -\frac{1}{c^2}\gamma^2\vec{v}\times\vec{E}_\perp + \frac{1}{c^2}\gamma^2\vec{v}\times\vec{E}_\perp = 0.$$

(d) $\vec{E}'' = \vec{E}$ and $\vec{B}'' = \vec{B}$ because the unprimed and double-primed frames are the same frame ($\vec{v} - \vec{v} = \vec{0}$ is their relative velocity).

magnetic dipoles

1. $\vec{V} = V_x \hat{x} + V_y \hat{y} + V_z \hat{z}$

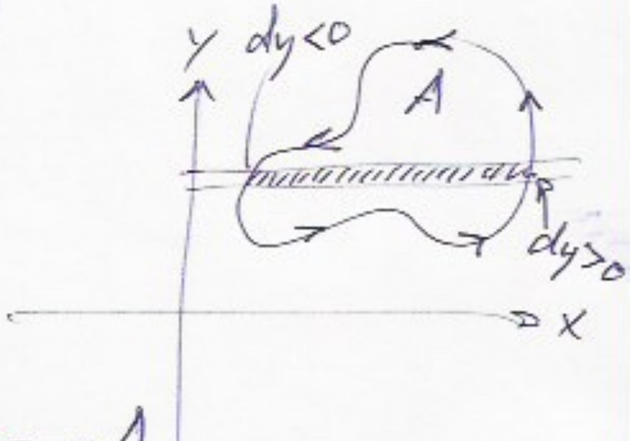
$$\oint_C (\vec{r}\cdot\vec{V})\, d\vec{r} = \oint_C (xV_x + yV_y)(dx\,\hat{x} + dy\,\hat{y})$$

(since $\vec{r}$ & $d\vec{r}$ are in the x-y plane)

$V_x, V_y, \hat{x}, \hat{y}$ are constants

$$\oint_C x\,dx = \oint_C \tfrac{1}{2} d(x^2) = 0$$

$$\oint_C y\,dy = \oint_C \tfrac{1}{2} d(y^2) = 0$$



$$\oint_C x\,dy = \text{area enclosed by } C = A$$

$$\oint_C y\,dx = \oint_C \left(d(xy) - x\,dy\right) = -A$$

$$\rightarrow \oint_C (\vec{r}\cdot\vec{V})\, d\vec{r} = V_x A \hat{y} - V_y A \hat{x}$$

$$= (A\hat{z}) \times (V_x \hat{x} + V_y \hat{y} + V_z \hat{z})$$

2. $\vec{F} = \oint_C d\vec{F} = I\underbrace{\left(\oint_C d\vec{r}\right)}_{0} \times \underbrace{\vec{B}}_{\text{constant}}$

$= 0$

$$\vec{T} = I\oint_C \left(\vec{r}\times\underbrace{(d\vec{r}\times\vec{B})}\right)$$

$$\underset{①}{(\vec{r}\cdot\vec{B})\,d\vec{r}} - \underset{②}{(\vec{r}\cdot d\vec{r})\,\vec{B}}$$

① is an instance of the integral of part 1. with $\vec{V} = \vec{B}$

$\Rightarrow I\vec{A}\times\vec{B}$

② $\oint_C \vec{r}\cdot d\vec{r} = \oint_C d\left(\tfrac{1}{2}\vec{r}\cdot\vec{r}\right) = 0$

$\vec{T} = I\vec{A}\times\vec{B}\,, \quad \vec{m} \equiv I\vec{A}$

3. $$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi} \oint_C \left(\frac{1}{r} + \frac{1}{r^2}\hat{r}\cdot\vec{r}\,' + \cdots \right) d\vec{r}\,'$$

constants

$$\oint_C d\vec{r}\,' = 0$$

$$\oint_C (\hat{r}\cdot\vec{r}\,')\,d\vec{r}\,' = \vec{A}\times\hat{r} \quad \text{(by part 1)}$$

$\vec{m}$

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\cdot\frac{1}{r^2}\,\vec{A}\times\hat{r}$$

$$= \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\hat{r}}{r^2}$$

(don't confuse the vector potential $\vec{A}$ with the loop's area vector $\vec{A}$!)