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Muon-catalyzed fusion

Introduction

In 1957, Alvarez was at Berkeley's liquid hydrogen bubble chamber observing the tracks left by cosmic ray kaons when he found the surprising track pattern in Figure 1 [1]. A muon traced a curve (beginning at point A in the figure) in the laboratory **B**-field (tighter than the kaons, background) before coming to rest bound to a hydrogen atom (B). The track disappeared for a time while the muonic hydrogen atom drifted along until, suddenly, the muon shot off again, apparently at an appreciable fraction of lightspeed (C, finally decaying at D). Alvarez was able to correctly explain his result as follows.

The fundamental scales of length and energy in muonic atoms and molecules (those having their electrons replaced by muons) are given by

$$a_{\mathrm{B},\mu} = \frac{\hbar^2}{m_\mu e^2} = \frac{m_e}{m_\mu} a_{\mathrm{B},\mu} = 2.6 \times 10^{-11}\mathrm{cm} \quad E_{\mathrm{H},\mu} = \frac{e^2}{a_{\mathrm{B},\mu}} = \frac{m_\mu}{m_e} E_H = 5500\mathrm{eV}$$

so that muonic atoms and molecules are about $m_\mu/m_e \approx 207$ times smaller and more tightly bound than their electronic counterparts. In a muonic hydrogen molecule, this reduction in scale leads to an appreciable probability for the nuclei to tunnel past their mutual Coulomb repulsion (the probability falls about exponentially with the tunneled distance) into the regime of the strong interaction. The resulting nuclear fusion reaction (proton-deuterium fusion, in Alvarez' case) will release a great deal of energy and eject the muon toward a neighboring hydrogen molecule, where the process repeats. The muon thus acts as a catalyst for many fusion reactions in a cyclic process.

Alvarez' observation suggested the exciting possibility of energy generation by nuclear fusion at pedestrian temperature and pressure. Unfortunately, Jackson's 1957 comprehensive analysis [2] demonstrated that the muon catalyzes too few fusion reactions to repay the energetic debt of 9GeV for its production using a particle accelerator, ruling out muon-catalyzed fusion as an energy source. Jackson found two factors limiting the number of fusion: first, the finite overall reaction rate Γ_{cycle} limits the number of fusions within the lifetime $\tau_\mu = 2.2 \times 10^{-6}$s of the muon. Secondly, the muon has a possibility of permanently binding to one of the fusion products, removing it from the catalytic cycle.

We will examine each of the steps of the muon-catalyzed fusion cycle to see how the first of the obstacles turned out to be illusory, although the second obstacle still remains.

The muonic hydrogen atom

The muon catalysis cycle begins with a free muon incident on a mixture of liquid deuterium and tritium molecules. This could be a fresh muon generated by cosmic rays or an accelerator-based source, or a muon liberated after catalyzing a fusion reaction, in which case it will be traveling with about 2keV energy (one Bohr velocity $\alpha_{fs}c$). The muon will lose energy in ionizing collisions with hydrogen atoms in the target until it has only about one ionization energy left ($\approx 10eV$), at which point it will be captured into a bound orbit at about the same radius as the last electron it kicked off. The muon will cascade down from this high energy level $n \approx \sqrt{m_\mu/m_e}$ down to the $n = 1$ level, emitting a series of X-rays to release the binding energy $\frac{1}{2}E_{H\mu} = 2300$eV. [3]

Supposing the muon has been captured around a deuterium nucleus to form a $d\mu$ muonic atom, it will be transferred to a tritium nucleus at the first opportunity. The transferal is energetically favorable because of the higher reduced mass of the $t\mu$ system as compared to the $d\mu$ system; the binding to tritium in the $1s$ state is stronger by about $\frac{1}{2}E_{\mathrm{H},\mu}\sqrt{1/m_d - 1/m_t} \approx 50$eV.

For the initial capture, the cascade to the ground state, and transferal to tritium, the reaction rates are roughly $\Gamma_{\mathrm{capture}} \approx 10^{13}\mathrm{s}^{-1}$, $\Gamma_{\mathrm{cascade}} \approx 10^{11}\mathrm{s}^{-1}$, and $\Gamma_{\mathrm{transfer}} \approx 3 \times 10^{8}\mathrm{s}^{-1}$ [3]. Only the transferal to tritium has a reaction rate slow enough to affect the overall catalysis cycle rate.

The muonic hydrogen molecule

At this point, the muon has been captured in the ground state of a muonic tritium atom $t\mu$ surrounded by ordinary, electronic deuterium molecules D_2. In order to catalyze a deuterium-tritium fusion reaction, the muon must first join two nuclei in a tightly bound state $(dt\mu)^+$: a muonic hydrogen molecule.

Because the $t\mu$ atom appears on the electronic length scale as a tiny ($\sim a_{B,\mu}$), neutral particle, it easily lumbers inside the electron cloud of a D_2 molecule and binds with the d nucleus to form a $(dt\mu)^+$ ion. The binding energy released (on the muonic energy scale $E_{\mathrm{H},\mu}$) ejects one of the two electrons of D_2 (an Auger decay). This reaction

$$t\mu + \mathrm{D}_2 \longrightarrow [(dt\mu)de]^+ + e(\sim 1\mathrm{keV})$$

was estimated by Zeldovich in 1960 [4] to occur at a rate of $\Gamma_{\mathrm{molec,Auger}} \sim 10^6 \mathrm{s}^{-1}$. According to this theory, the muon would then only have time to catalyze one or two fusions before its decay.

Dzhelepov et al. had been conducting experiments on muon-catalyzed dd fusion when in 1966 their data revealed a surprising result: their formation rate of the similar $(dd\mu)^+$ molecule $\Gamma_{\mathrm{molec}} \sim 10^8 \mathrm{s}^{-1}$ was much faster than what was previously calculated and furthermore depended strongly on the temperature, ruling out the Auger decay mechanism just described [5].

In 1967 Vesman [6] published a theory of the muonic molecular formation process that fit the experimental data. He posited that the muonic molecule first forms in a weakly-bound excited state with the rotational and vibrational quantum numbers $|J,\nu\rangle = |1,1\rangle$ rather than the very tightly bound $|0,0\rangle$ state assumed previously. He argued that the small released binding energy ϵ_{b}, plus the kinetic energy of the reactants incident at thermal velocities $\epsilon_{\mathrm{therm}}$, might lie within the range of energies that could be absorbed by the rotational-vibrational energy spectrum of the full electronic molecule $\Delta\epsilon_{\mathrm{rot,vib}}$ (i.e., the energy would be taken up by motion of the second deuterium nucleus relative to the muonic ion $(dt\mu)^+$). Then the reaction

$$t\mu + (\mathrm{D}_2)_{|0,0\rangle} \rightleftharpoons [(dt\mu)_{|1,1\rangle} dee]_{|K,n_v\rangle}$$

would be fast only for reactant particles that were incident at resonant energies $\epsilon_{\mathrm{therm}}$ such that

$$\epsilon_{\mathrm{b}} + \epsilon_{\mathrm{therm}} = \Delta\epsilon_{\mathrm{rot,vib}}$$

Since the populations of particles moving exactly at the resonant energies have a strong temperature dependence through the Maxwell-Boltzmann distribution $f(\epsilon_{\mathrm{therm}}; T)$, Vesman's theory thus explained the observed temperature dependence.

We can estimate the binding energy of the $(dt\mu)^+$ state required for Vesman's theory to work. The energy scale for vibrational transitions of electronic molecules is $\sqrt{m_e/M_N}E_H \approx 0.4\mathrm{eV}$ for D_2 [7], while the energy scale of the Maxwell-Boltzmann distribution is much smaller, $k_B T < 0.01\mathrm{eV}$ at room temperature and below. Thus the binding energy ϵ_{b} should be $\sim 1\mathrm{eV}$.

Demonstrating the existence of the crucial weakly-bound $|J,\nu\rangle = |1,1\rangle$ state of $(dt\mu)^+$ proved to be a formidable theoretical challenge because the muonic and nuclear motions lacked the separation of scales characteristic of electronic molecules [3]. A first calculation by Ponomarev of the binding energy using the Born-Oppenheimer approximation yielded a large binding energy $\sim$ 10eV that seemed to rule out Vesman's hypothesized state entirely. The obvious next step of applying higher-order corrections to the Born-Oppenheimer approximation implied no bound state existed at all. Gershtein and Ponomarev worked on the problem for a decade, arriving in 1977 at a proof that this bound state indeed exists and has energy $\approx -1\mathrm{eV}$. In the 1980s Hu performed variational calculations and fixed the binding energy at 0.66eV [8]. Once the wavefunction and energy of the $|1,1\rangle$ state were known, it was possible to estimate the rate for this process to occur using Fermi's golden rule [7][3]

$$\Gamma_{\mathrm{molec,Vesman}} = N_0 \sum_{K,n_\nu} \int_0^\infty \mathrm{d}\epsilon f(\epsilon; T) 2\pi\delta((\epsilon + \epsilon_b) - \Delta\epsilon_{K,n_\nu})|\langle\psi_0|H'|\psi_f\rangle|^2$$

Where N_0 is the density of the target, f is the Maxwell distribution of incident $t\mu$ energies, and we sum over the final rotational-vibrational states $|K,n_\nu\rangle$ of the full system $[(dt\mu)_{|1,1\rangle} dee]$ with their various energies $\Delta\epsilon_f$. The initial state ψ_0 is taken as a product of (1) the incident $t\mu$ atom in its 1s state times a plane wave factor to represent its motion (2) one of the two d nuclei colocated with the t nucleus. A transition to the

final state ψ_f (the theoretical wavefunction for the $|1,1\rangle$ state) is expected to occur under the influence of a time-dependent perturbation

$$\hat{H}' = -\mathbf{E} \cdot \sum_{\alpha} e_\alpha (\mathbf{r}_\alpha - \mathbf{r}_{\rm cm})$$

due to the electric field $\mathbf{E}$ of the second d nucleus and two electrons of the D_2 molecule interacting with the dipole moment of the $dt\mu$ system with a center of mass $\mathbf{r}_{\rm cm}$. The classical interpretation is that the electric field of the other constituents of the deuterium molecule applies a torque to the $dt\mu$ subsystem to spin it up into its $J = 1$ rotational state from the original $J = 0$ spherically symmetric state.

The $dt\mu$ molecular formation rate has been calculated at $\Gamma_{\rm molec,Vesman} \approx 4 \times 10^8 {\rm s}^{-1}$. This remains the rate-limiting step, but now the muon has time to catalyze 500 or more fusions (neglecting the sticking loss mechanism, to be discussed later) within its lifetime, reopening the exciting possibility of practical energy production.

Fusion occurs only very slowly in the $|1,1\rangle$ state of $dt\mu$ because of the centrifugal barrier, so an Auger decay must first occur to the $|0,0\rangle$ state where fusion occurs most rapidly, with an electron accepting the large binding energy of the ground state:

$$t\mu + (\mathrm{D}_2)_{|0,0\rangle} \rightleftharpoons [(dt\mu)_{|1,1\rangle} dee]_{|K,n_v\rangle} \rightarrow [(dt\mu)_{|0,0\rangle} de]^+_{|K,n_v\rangle} + e(320\mathrm{eV})$$

This particular Auger decay is quite fast ($\sim 10^{12}{\rm s}^{-1}$), meaning that the reverse Vesman process that would otherwise break up the weakly-bound state at a rate of ($\sim 10^8{\rm s}^{-1}$) is no competition. The rate calculation for this Auger decay is similar to the calculation above. The hydrogenic atom $dt\mu e$ experiences a time-dependent perturbation from the $dt\mu$'s dipole interacting with the electron's $\mathbf{E}$-field, causing the electron to transition from a $1s$ bound state to a plane-wave state. [5]

Nuclear fusion rate

Nuclear fusion of the deuterium and tritium nuclei occurs soon after the formation of the muonic molecular ion $(dt\mu)^+$ by the reaction

$$dt\mu \rightarrow \mu + \alpha + n + 13.7\mathrm{MeV}$$

We can estimate the typical time-to-fusion $\Gamma^{-1}_{\rm fus}$ using the same approach as Jackson and compare our result to both detailed calculations and experiment. We proceed by means of the Born-Oppenheimer approximation, letting the nuclear slow subsystem slowly evolve under the influence of an energy surface generated by the muonic fast subsystem. Note that the accuracy of this approximation is lost in the transcription from electronic to muonic molecules because the nonadiabatic terms are $\sim \sqrt{m_{\rm fast}/M_{\rm slow}}$. This parameter is ~ 100 for electronic but only ~ 10 for muonic systems due to the muon's higher mass, but we use the approximation regardless for this estimate.

The muonic subsystem's Hamiltonian (in muonic units)

$$\hat{H}_\mu = -\frac{1}{2}\nabla^2 - \frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{R}$$

combines the muon's kinetic energy, its attraction to the two nuclei, and the nuclear repulsion. A variational method gives an approximate ground state energy with the nuclear separation R as a parameter

$$V(R) = -\frac{1.5}{(0.127R+1)^4} + \frac{1}{R}$$

The energy surface (solid curve in Figure 2, above) has a minimum V_0 at a nuclear separation R_0 allowing for the stability of $(dt\mu)^+$.

The radial nuclear Hamiltonian found by the substitution $\Psi(\mathbf{R}) = Y_{lm}(\theta,\phi)u(R)/R$ is

$$\hat{H}_{dt} = -\frac{1}{2M}\frac{d^2}{dR^2} + \frac{(J+\frac{1}{2})^2}{2MR^2} + V(R)$$

where M is the reduced mass of the two nuclei and J is the total rotational momentum number. The substitution $J(J+1) \rightarrow (J+1/2)^2$ is made because we intend to apply the semiclassical approximation to

radial problems (Jackson's wizardry again). Consider only the rotational ground state $J = 0$ because, for $J \geq 1$, the centrifugal potential barrier leads to a much slower fusion rate. A rotational transition $J = 1 \to 0$ will occur long before fusion does.

In the vibrational ground state $\nu = 0$, the nuclei will be well-localized near the equilibrium separation R_0, so we may approximate the muonic effective potential by a fitted harmonic oscillator potential

$$V(R) \approx V_0 + \frac{1}{2}M\omega^2(R - R_0)^2 \quad M\omega^2 = \left[\frac{d^2V}{dR^2}\right]_{R_0}$$

The vibrational ground state has an energy of $V = V_0 + \frac{1}{2}\omega$ and a Gaussian wavefunction

$$u(R) = \left(\frac{M\omega}{\pi}\right)^{1/4} \exp\left(\frac{-M\omega(R - R_0)^2}{2}\right)$$

which should not be extrapolated beyond the inner classical turning point R_1 because of the departure of $V(R) \sim 1/R$ from harmonicity there.

Instead, we employ the WKB approximation to connect the harmonic oscillator wavefunction near the potential minimum to a semiclassical decaying exponential wavefunction inside the turning point using the connection formula

$$u(R) = \frac{2B}{\sqrt{|p(R)|}} \cos\left(\int_{R_1}^{R} dR p(R) - \frac{\pi}{4}\right) \Longrightarrow u(R) = \frac{B}{\sqrt{|p(R)|}} \exp\left(-\int_{R}^{R_1} dR|p(R)|\right)$$

where

$$p^2(R) = 2M\left(E - V(R)\right) - \frac{1}{4R^2}$$

is the classical momentum. We set the amplitude B such that the semiclassical wavefunction and the harmonic oscillator coincide at R_0. The harmonic-oscillator and WKB wavefunctions are plotted in Figure 2 as dashed and solid lines, respectively. To find the behavior of the wavefunction for small nuclear separations, observe that $|p| \to 1/(2R)$ as $R \to 0$. Then we can break up the integrand

$$u(R) = \frac{B}{\sqrt{|p(R)|}} \exp\left[-\int_{R}^{R_1} dR\left(\frac{1}{2R}\right) - \int_{R}^{R_1} dR\left(|p(R)| - \frac{1}{2R}\right)\right] \to BR\exp\left(-\frac{\lambda(R)}{2}\right)$$

to find that $u(R)$ varies linearly with small R so that a finite value of the wavefunction $\Phi(R) = u(R)/R$ exists, where the tunneling factor

$$\lambda(R) = \ln\frac{R_1}{2} - \int_{R}^{R_1} dR\left(2|p(R)| - \frac{1}{R}\right)$$

converges and can be computed numerically.

The fusion rate is proportional to the probability density at zero nuclear separation

$$\Gamma_{\text{fus}} = A\lim_{R\to 0}|\Psi(\mathbf{R})|^2 = A\frac{B^2}{4\pi}\exp(-\lambda(0^+))$$

with the $1/4\pi$ coming from the $J = 0$ angular wavefunction. The reaction constant A can be found by extrapolating fusion cross section data measured at keV beam energies down to the low incident energies relevant to this problem using the Gamow formula. The low-energy cross section from [9] for dt fusion is $A_{dt} = 2.0 \times 10^{-14}\text{cm}^3/\text{s}$ from which we calculate $\Gamma_{\text{fus},dt} = 1.1 \times 10^{12}\text{s}^{-1}$.

As a sanity check, if we substitute the muon for an electron, the length R_1 of the Coulomb barrier scales up while the classical momentum $|p|$ scales down, so that the tunneling factor $\lambda(0^+)$ increases by $\sqrt{m_\mu/m_e}$. [2] Then the fusion probability will be $\exp((\sqrt{m_\mu/m_e} - 1)\lambda_\mu) \sim 10^{57}$ times smaller than for muonic systems - tough odds.

Muon sticking

Muon sticking represents the most important loss mechanism for muons in the catalysis cycle. Following the fusion reaction, the muon has a small probability to "stick" to the α particle in a bound state, the initial sticking probability P^0_{stick}. The binding energy is quite high ($\frac{1}{2}Z^2E_{H\mu} \approx 10$keV in the 1s state), so that the muon is usually carried off with the α and cannot catalyze any further fusion reactions. There is an appreciable "reactivation" probability R for the muon to be stripped off the α in collisions with other atoms, so that the effective sticking probability is

$$P_{\text{stick}} = P^0_{\text{stick}}(1 - R)$$

We follow Jackson's approach [2] here to make a simple estimate of P^0_{stick}. We assume the muonic wavefunction responds adiabatically to the slow nuclear motion so that, immediately before fusion, the muon lies in the $1s$ state of the unstable dt compound nucleus. The fusion reaction releases 17.6MeV of energy, of which the α takes up 3.6MeV, travelling at $0.03c$. Immediately after fusion, the muon can either stick to the α or be left in its wake. The muon's final state if it sticks is taken to be the $1s$ state of $^4\text{He}^+$ times a plane wave factor $\exp(-iv_\alpha\hat{\mathbf{z}}\cdot\mathbf{r})$ because we have boosted the stationary state to the α's frame. Strictly speaking the contribution from final states $2s$ and higher should be included, but these states are more easily reactivated in collisions (77% of the sticking occurs in the $1s$ state for this reason [3]).

If sticking occurs, the muon's wavefunction thus makes the transition

$$\psi_0(\mathbf{r}) = \frac{1}{\sqrt{\pi}}e^{-r} \longrightarrow \psi_f(\mathbf{r}) = \psi_0(\mathbf{r})\exp(-iv_\alpha\hat{\mathbf{z}}\cdot\mathbf{r})$$

The α completes its acceleration within a length of the order of the nuclear size $r_\alpha \sim 10^{-13}$cm and a time of the order $\tau \sim r_\alpha/v_\alpha \sim 10^{-22}$s which are both negligibly small compared to the muonic length and time scales $a_B \sim 10^{-11}$cm and $h/E_H \sim 10^{-18}$s. Therefore, the sudden approximation can be used to calculate the probability of transition to the stuck state as an overlap integral:

$$P^0_{\text{stick}} = \left|\int d^3\mathbf{r}\psi_f^*(\mathbf{r})\psi_0(\mathbf{r})\right|^2 \approx \left[1 + \left(\frac{v_\alpha}{4}\right)^2\right]^{-4}$$

Where the final expression comes from Jackson (questionably) expanding the plane wave factor to first order. The α travels at $v_\alpha = 6.0$ (in units of the Bohr velocity c/α) for this reaction, leading to an initial sticking probability of 0.9%.

The Born-Oppenheimer assumption for the initial wavefunction turns out to be a poor one here, in the cases where nuclear fusion occurs, the two nuclei must approach rapidly to the point of coalescence [10]. The most recent detailed calculation by Hu [8] of the initial sticking probability using variational wavefunctions gives 0.89% (Jackson's result was correct by a coincidence of factors). The reactivation rate R is expected to by around 30% at liquid hydrogen densities for an overall sticking rate of 0.62%, limiting the number of fusions to $1/P_{\text{stick}} = 160$ per muon.

Each muon thus liberates 2.4GeV of energy, far exceeding its 106MeV rest mast but still impractical for energy generation given the 9GeV debt to produce each muon. Regardless of practical applications, the muon-catalyzed fusion cycle brings together nuclear, muonic, and electronic effects to yield some fascinating physical phenomena.

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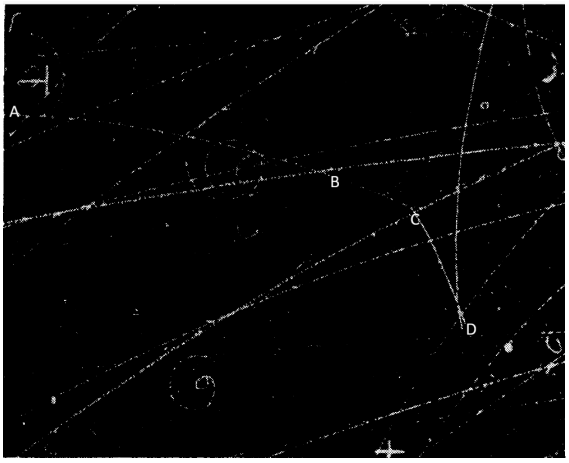

Figure 1: The muon-catalyzed fusion event observed by Alvarez in 1957.

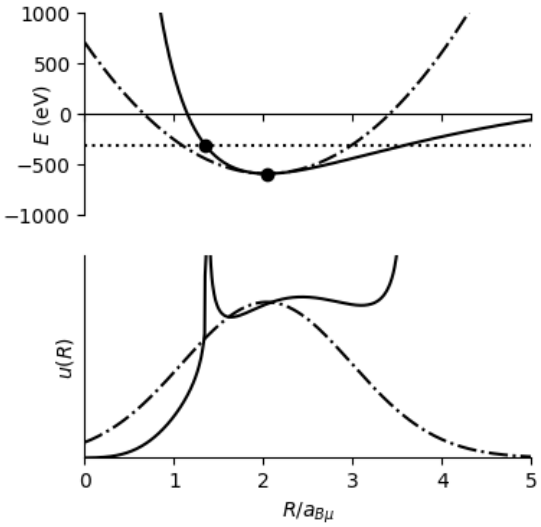

Figure 2: Potentials and wavefunctions used in the estimate of the nuclear fusion rate. Above: exact (solid) and fitted harmonic (dashed) potentials. Below: semiclassical (solid) and harmonic oscillator (dashed) wavefunctions.

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