Assignment 4 Solutions

Electric field inside an atom

We are given the volume charge density as a function of the distance from the origin. For the total charge in the electron cloud to be $-Q$ ($-e$ in the question statement, but I'm using $-Q$ to avoid confusion with the exponential), we need the integral of the volume charge density over all space to equal $-Q$.

$$\int \rho(r) dV = -Q$$
$$\Rightarrow A \int_0^\infty 4\pi r^2 e^{-r/a} dr = -Q$$
$$\Rightarrow 8\pi A a^3 = -Q$$
$$\Rightarrow A = -\frac{Q}{8\pi a^3}$$

Next, we need to find the electric field inside the hydrogen atom. Since the charge distribution is spherically symmetric, the electric field at any point will only be a function of the distance of the point from the origin. We will use the integral form of Gauss' law and take the Gaussian surface as a sphere of radius r. The charge enclosed in the Gaussian surface,

$$\begin{aligned} q_{enc} &= Q + \int_0^r \rho(r') 4\pi r'^2 dr' \\ &= Q e^{-r/a} \left(1 + \frac{r}{a} + \frac{1}{2}\left(\frac{r}{a}\right)^2\right). \end{aligned}$$

$$\begin{aligned} \int \vec{E} \cdot d\vec{A} &= \frac{q_{enc}}{\epsilon_0} \\ 4\pi r^2 E_r &= \frac{Q e^{-r/a} \left(1 + \frac{r}{a} + \frac{1}{2}\left(\frac{r}{a}\right)^2\right)}{\epsilon_0} \\ E_r &= \frac{Q e^{-r/a} \left(1 + \frac{r}{a} + \frac{1}{2}\left(\frac{r}{a}\right)^2\right)}{4\pi r^2 \epsilon_0} \end{aligned}$$

and

$$\vec{E} = E_r \hat{r}$$

To find the form of the electric field for the limiting case $r << a$, let us Taylor expand the exponential.

$$\begin{aligned} E_r &\approx \frac{Q}{4\pi r^2 \epsilon_0}\left(1 - \frac{r}{a} + \frac{1}{2}\left(\frac{r}{a}\right)^2\right)\left(1 + \frac{r}{a} + \frac{1}{2}\left(\frac{r}{a}\right)^2\right) \\ &\approx \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r^2} + \mathcal{O}\left(r^2/a^2\right)\right) \end{aligned}$$

Thus, for $r \ll a$, the leading term goes as $1/r^2$, which is similar to that of a point charge. This is what you would expect near the nucleus of the hydrogen atom.

For the limit $r \gg a$, the electric field falls off as $e^{-r/a}$. Thus, it takes the form

$$\begin{aligned} E_r &\approx \frac{Qe^{-r/a}}{4\pi r^2 \epsilon_0}\frac{1}{2}\left(\frac{r}{a}\right)^2 \\ &= \frac{Qe^{-r/a}}{8\pi a^2 \epsilon_0} \end{aligned}$$

Point charge in a uniform field

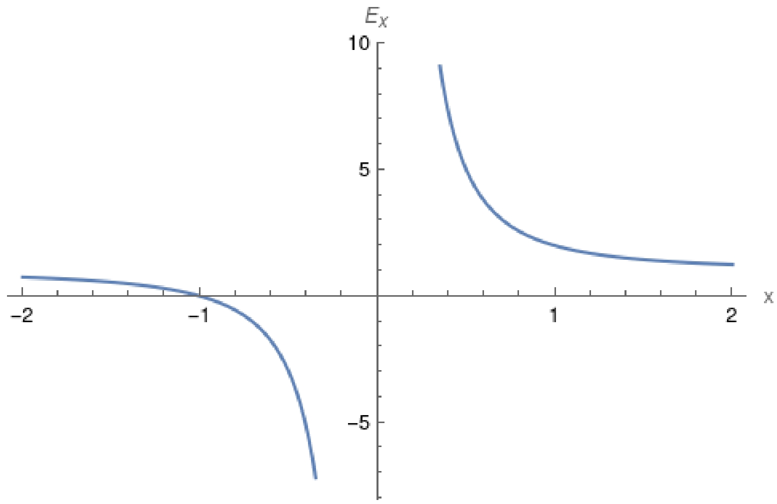
The net electric field due to the uniform electric field and the point charge is given by,

$$\begin{aligned} \vec{E} &= E_0\hat{x} + \frac{kq}{r^2}\hat{r} \\ &= \left(E_0 + \frac{kqx}{(x^2+y^2+z^2)^{3/2}}\right)\hat{x} + \frac{kqy}{(x^2+y^2+z^2)^{3/2}}\hat{y} + \frac{kqz}{(x^2+y^2+z^2)^{3/2}}\hat{z} \end{aligned}$$

Along the x-axis, the total electric field takes the form

$$\begin{aligned} E_x(x) &= E_0 + \frac{kq}{x^2}, \quad x > 0 \\ &= E_0 - \frac{kq}{x^2}, \quad x < 0 \end{aligned}$$

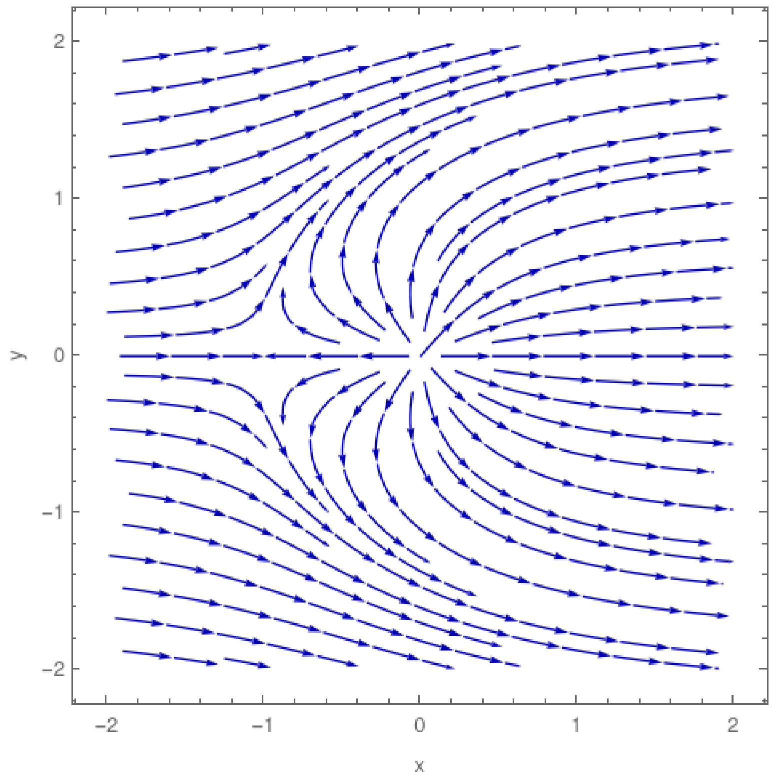
Below is the figure with $kq, E_0 = 1$.



In the xy plane,

$$\vec{E}(x,y) = \left(E_0 + \frac{kqx}{(x^2+y^2)^{3/2}}\right)\hat{x} + \frac{kqy}{(x^2+y^2)^{3/2}}\hat{y}$$

In the xy plane, the field lines associated with $\vec{E}(\vec{r})$ look like the figure below $(E_0, kq = 1)$:



Let $p = (-x_p, 0, 0)$ with $x_p > 0$ be the point on the x-axis such that $\vec{E}(\vec{r}) = 0$. Then,

$$E_0 - \frac{kq}{x_p^2} = 0$$
$$\Rightarrow x_p = \sqrt{\frac{kq}{E_0}}$$

Now let us calculate the partial derivatives at p.

$$\left.\frac{\partial E_x}{\partial x}\right|_p = \left.\frac{-3kqx^2}{(x^2+y^2+z^2)^{5/2}} + \frac{kq}{(x^2+y^2+z^2)^{3/2}}\right|_{(-x_p,0,0)} = -2\frac{E_0^{3/2}}{(kq)^{1/2}}$$
$$\left.\frac{\partial E_y}{\partial y}\right|_p = \frac{E_0^{3/2}}{(kq)^{1/2}}$$
$$\left.\frac{\partial E_z}{\partial z}\right|_p = \frac{E_0^{3/2}}{(kq)^{1/2}}$$

The divergence of $\vec{E}$,

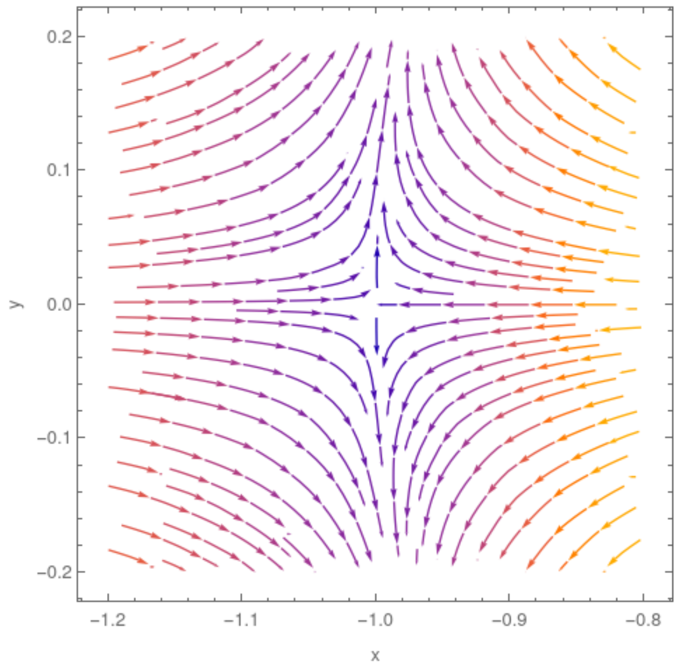
$$\vec{\nabla} \cdot \vec{E}|_p = \left.\frac{\partial E_x}{\partial x}\right|_p + \left.\frac{\partial E_y}{\partial y}\right|_p + \left.\frac{\partial E_z}{\partial z}\right|_p = 0$$

This satisfies the local form of Gauss' law $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$ as there is no charge density at the point p.

Consider a slight perturbation in the x-direction i.e. $-x_p \rightarrow -x_p + \delta x$. Then the x-component of the electric field becomes

$$E'_x = E_0 - \frac{kq}{(-x_p + \delta x)^2} < 0$$

Therefore, E'_x points to the left–towards the equilibrium point. If we had displaced it in the negative x-direction, we would get $E'_x > 0$, which points to the right–towards the equilibrium point, again.
Now let us consider displacements in y and z directions. In this case, the net electric field points away from the equilibrium point. This can be seen from the field lines near the point p. The field lines below are plotted in the xy plane with $E_0, kq = 1$. As you can see, the field lines along the x-axis point towards the equilibrium point, but the field lines along the y-axis point away from the equilibrium point. So a slight perturbation in any direction other than the x-axis would result in the particle following the field line away from the equilibrium point. The case for perturbations in the z-direction is similar to that in the y-direction.



While the x component of the force, $q\vec{E}$ points towards the equilibrium point for small displacements, the y and the z components point away from the equilibrium point. Therefore, the equilibrium is unstable.