Due date: Tuesday, December 10

This is the final assignment! You are able to, and should waste no time getting started on, the Hadamard model simulations because they might take some time to get right. For the other problems you are filling in details of the Potts model renormalization group calculation, most of which will be done in class. The first of these you are ready to do now but for the others you will have to wait until we arrive at the corresponding point of the calculation.

1. One of the most remarkable features of statistical mechanics is that energy measurements may be used to obtain purely entropic information. The fundamental relationship that enables this is the heat capacity integral:

$$s(\beta_2) - s(\beta_1) = -\int_{\beta_1}^{\beta_2} c(\beta)\frac{d\beta}{\beta}.$$

By measuring energy fluctuations (c) over a range of temperatures, one can infer the entropy difference of the system at the endpoints of that range. In this problem you apply this principle to the Hadamard model to approximately count the number of 8×8 Hadamard matrices.

In the Hadamard model, $s(0)$ is the logarithm of the volume of the space $\mathrm{SO}(n)$, the $n \times n$ orthogonal matrices of determinant 1, divided by n^2. Writing a general element $U \in \mathrm{SO}(n)$ as

$$U = e^X,$$

where X is a real anti-symmetric matrix, the volume element near the identity element is conventionally defined as[1]

$$d\mu = \prod_{1 \le i < j \le n} \sqrt{2}\, dX_{ij}.$$

With this convention[2],

$$\mathrm{vol}(\mathrm{SO}(n)) = 2^{(n-1)(n/4+1)} \prod_{k=2}^{n} \frac{\pi^{k/2}}{\Gamma(k/2)}.$$

[1]Consider the length of the line element in $\mathrm{SO}(n)$ when only X_{12} is varied, but having the effect of changing both U_{12} and U_{21}. Recall that the scale of the volume element in classical statistical mechanics is arbitrary and has no effect on entropy differences.

[2]Thanks to Junkai Dong for tracking this down in the literature!

Setting $n = 8$, taking the logarithm, and dividing by n^2, we obtain

$$s(0) = 0.48029817045547823$$

Now consider the opposite limit, $\beta \to \infty$. The matrix U will then be near one of the Hadamard-matrix minima of the Hamiltonian H. Let U^* be one such minimum — a Hadamard matrix divided by $\sqrt{n}$. Parameterize the matrices in the neighborhood as we did above:

$$U(X) = U^* e^X.$$

Show that for small X

$$H(U(X)) = -n^2 + \frac{n}{2}\,\mathrm{Tr}(XX^T) + \cdots,$$

which is remarkable in being independent of U^*. [*Hint:* The action of the absolute value in the Hamiltonian, in the neighborhood of the energy minimum, is the same as component-wise multiplication by U^*.]

Using the same measure $d\mu$ we used to define $s(0)$, show that for $\beta \to \infty$

$$Z(\beta) \sim \#(n) \left(\frac{2\pi}{\beta n}\right)^{n(n-1)/4},$$

where $\#(n)$ is the number of Hadamard matrices of order n (we assume $\#(n) > 0$). Finally, using the general relation

$$S(\beta) = \log Z(\beta) + \beta\langle H\rangle,$$

and remembering to divide by n^2, show that the Hadamard model has the following "specific" entropy in the limit of low temperature:

$$s(\beta) \sim \left(\frac{1 - 1/n}{4}\right)\log\left(\frac{2\pi e}{\beta n}\right) + \frac{1}{n^2}\log(\#(n)).$$

For the grand conclusion of this problem, modify your simulation code so that it performs the heat capacity integral from $\beta = 0$ to some low temperature, say $\beta = 10$, and compare the measured entropy difference to the analytic results above. For $n = 8$ you will not need too many sweeps for convergence, but you should vary the integration step $\Delta\beta$ to check convergence of the numerical integral. How closely can you recover $\#(8) = 2477260800$[3]?

[3]Caution: this is not an exponent but a footnote! The number is from OEIS A206711, after being careful to divide by 2 since we only want Hadamard matrices with determinant $+1$.

2. The adjacent lattice site term of $-\beta H$, for the Potts model Hamiltonian, may be written abstractly as $V^{\rm T} K V$, where V is a vector of $(q-1)|\mathcal{V}|$ components (corresponding to the $q-1$ vector components of the $\mathbf{v}(\mathbf{r})$'s at all lattice sites $\mathbf{r} \in \mathcal{V}$ in the system). The Hubbard-Stratonovich transformation converts the trace over the discrete variables V into a trace over continuous fields Ψ in one-to-one correspondence with the components of V. An "integral" part of the transformation is the evaluation of the associated coupling of the fields, $\Psi^{\rm T} K^{-1} \Psi$. Show that in the limit of slowly varying fields, so that the sum over sites may be approximated by an integral, one obtains

$$\Psi^{\rm T} K^{-1} \Psi \approx (\beta \epsilon D)^{-1} \int d^D\mathbf{r} \sum_{i=1}^{q-1} \left(\Psi_i^2 + \frac{1}{2D} |\nabla \Psi_i|^2 \right).$$

[*Hint:* The eigenvectors of the matrix K are plane waves in $\mathbf{r}$ with constant polarization in the $(q-1)$-space of the Potts model vectors $\mathbf{v}$. The action of K^{-1} on these eigenvectors is just to multiply the eigenvector by the inverse of the eigenvalue. Express the most general Ψ in terms of these eigenvectors and thereby obtain a formula for $\Psi^{\rm T} K^{-1} \Psi$.]

3. By closely mirroring the in-lecture calculation of $\langle \mathcal{H}'^2 \rangle_{\rm conn}$, that gave us $\delta c/\delta\lambda$ and $\delta r/\delta\lambda$, calculate $\langle \mathcal{H}'^3 \rangle_{\rm conn}$ and show that

$$\frac{\delta w}{\delta \lambda} = 36(q-3) w^3 \frac{\Omega_D K^D}{(K^2+r)^3}.$$

The following are some checks on parts of the calculation:

(a) There are only two Feynman diagrams to be considered. Draw both of them, not because they will help you do the calculation, but because it will impress your friends!

One of the diagrams can be neglected because there is no way for the three internal momentum variables (associated with the traced-out fields) to all be in the momentum shell while having the external momenta (associated with the un-traced fields) be small.

(b) The single Feynman diagram that contributes will have in it the factor

$$I(p,p') = \int^{>} \frac{d^D k}{(2\pi)^D} \int^{>} \frac{d^D k'}{(2\pi)^D} \int^{>} \frac{d^D k''}{(2\pi)^D} \frac{(2\pi)^D \delta^D(p - k'' + k)(2\pi)^D \delta^D(p' - k + k')}{(k^2 + r)(k'^2 + r)(k''^2 + r)},$$

where p, p' and $-p - p'$ are the three small external momenta. Since we are only interested in the renormalization of the low momentum limit of the cubic term, you only need to evaluate $I(0,0)$.

(c) The q-ology for the relevant Feynman diagram involves the sum (over repeated indices)

$$S_{iln} = Q_{ijk} Q_{klm} Q_{mjn}.$$

When you evaluate this sum you will find that, not surprisingly, it is proportional to the only permutation-invariant 3-index tensor at hand, Q_{iln}.

4. In terms of the rescaled parameters

$$\tilde{r} = r/K^2 \qquad \tilde{w} = w\sqrt{\Omega_D K^{D-6}}$$

the flow equations for $q = 1$ (percolation) take the form

$$\begin{aligned} \dot{\tilde{r}} &= 2\tilde{r} + 18\tilde{w}^2 - 30\tilde{r}\tilde{w}^2 \\ \dot{\tilde{w}} &= \frac{\epsilon}{2}\tilde{w} - 63\tilde{w}^3. \end{aligned}$$

(a) Find the fixed points of this system to lowest order in ϵ, both for $D > 6$ ($\epsilon < 0$) and $D < 6$ ($\epsilon > 0$). Do you find that the effects of fluctuations raise or lower the ordering temperature of the $q \to 1$ Potts model?

(b) Determine to lowest order in ϵ the eigenvectors and eigenvalues of the flow equations linearized about the fixed points.