Due date: Thursday, October 31

1. The Hadamard model has a surprisingly simple order parameter. If you haven't guessed it yet, here are two clues:

   - It is a property of individual matrix elements. That is, from the distribution of U_{ij}, for a single (i, j), you would be able to tell which phase the system is in.
   - As we've seen (empirically), with increasing n the transition moves to ever lower temperatures, and the energy approaches the ground state value, $-n^2$. However, if the low energy/temperature properties — on *both* sides of the transition — could be derived from the neighborhood of a *smooth* potential function, the heat capacity would be $n^2/4$ by equipartition. But, again from simulations, that is only true on the low temperature side of the transition.

   Add measurement of the order parameter to your Hadamard model simulator and run a range of β for $n = 16$ that straddles the transition to confirm that the order parameter tracks the thermodynamic behavior.

   For the planned publication it would be nice to have data up to $n = 24$. Estimate how many sweeps would be required (using Givens transitions) to obtain that data.

   If you wish to become a coauthor on the Hadamard-model paper, declare your interest in this assignment! Veit will write the first draft but could use some help locating relevant prior work. Specifically, we are interested in studies of continuum models *without* quenched disorder that address the question of an equilibrium glass phase. "Quenched disorder" means random parameters, as in spin-glass models. We hope to have a near-final draft of the manuscript by the end of the semester.

2. A very useful theorem in graph theory relates a combinatorial property of a graph, the number of spanning trees it contains, and a matrix property, the determinant of the graph's *Laplacian*. In this problem you will be working with the set of connected graphs more properly called *multigraphs*. In a multigraph there can be multiple edges between a pair of vertices. However, our graphs will not have *loops*, which are edges joining vertices to themselves. Let $\tau(C)$ be the number of spanning trees in the multigraph C. If C has n vertices, the graph Laplacian $\Delta(C)$ is the $n \times n$ matrix whose diagonal elements are the edge degrees of the vertices and whose off-diagonal elements are the negative counts of the number of edges between pairs of vertices. Finally, let $\Delta_i(C)$ be the $(n-1)\times(n-1)$ matrix obtained by deleting row and column i of $\Delta(C)$. Kirchhoff's *matrix-tree theorem*[1] states that (for any i)
$$\tau(C) = \det \Delta_i(C).$$

   (a) Let A be an $N \times N$ real symmetric positive-definite matrix (so that it may be written as $A = B^{\mathrm{T}} B$ where B is nonsingular). Show that
   $$\int d^N x \exp\left(-x^{\mathrm{T}} A x\right) = \sqrt{\frac{\pi^N}{\det A}}.$$

   (b) We are interested in the case where $N = (n-1)D$ and A is comprised of $D \times D$ identity blocks multiplied by elements of an $(n-1)\times(n-1)$ matrix Δ_1. Show that
   $$\det A = (\det \Delta_1)^D.$$
   [Hint: Compare the eigenvalues of A and Δ_1.]

[1] To prove the theorem (not required) you can use induction on the formula $\tau(C) = \tau(C - e) + \tau(C \circ e)$ and the corresponding statement for the graph Laplacians. Here $C - e$ is the graph C with edge e removed, while $C \circ e$ is the graph obtained by merging the two vertices connected by e.

(c) Low density expansions in statistical mechanics sometimes involve integrals of the form

$$I(C) = \int d^D x_2 \cdots \int d^D x_n \exp\left(-\sum_{(ij)\in C} (x_i - x_j)^2\right), \tag{1}$$

where C is a connected graph of n vertices (not a multigraph in this case) and the sum runs over all edges (ij) in C. Using the previous parts of this problem, show that

$$I(C) = \left(\frac{\pi^{n-1}}{\det \Delta_1(C)}\right)^{D/2}.$$

[Hint: By translational invariance of the integrand it does not matter which coordinate is fixed and not integrated over (in equation (1) it is x_1). For the same reason, the value of the fixed coordinate does not matter so make the simplest choice: set it equal to zero.]

(d) The integral $I(C)$ is easy to evaluate when C is a tree. Perform the integral and show that $\det \Delta_1(C) = 1$ when C is a tree, a simple check of the matrix-tree theorem.

(e) The complete graph K_n is the graph on n vertices where all pairs of edges are joined by one edge. Using the matrix-tree theorem prove Cayley's formula:

$$\tau(K_n) = n^{n-2}.$$

Recall that we made use of this fact in the RGM.