Due date: Thursday, October 10

You should get started on the first problem, about the Hadamard model, as soon as possible because the simulations might take whole days. For this problem and future Hadamard simulations you may work in teams, but team members should write up separate reports in their homework solutions.

1. Customize your Hadamard-model simulator — or one of the contributed simulators from the course Github site — to address a number of outstanding features of the model. At a minimum you want to be able to specify, for one run: n, the range and increment for β, and the number of sweeps in the average at each β. The output should list the energy e and heat capacity c at each β, along with estimates of the errors. Here are the simulations for this assignment:

   - For $n = 12$ make plots of $e(\beta)$ and $c(\beta)$ for β between 0 and 10, with sufficiently small increments to resolve the abrupt drop in energy and see the heat capacity peak as a hill rather than a single point. Check that $e(\beta)$ and $c(\beta)$ agree with the high temperature limit from the previous assignment and the equipartition forms $e(\beta) \sim -1 + 1/(4\beta)$, $c(\beta) \sim 1/4$ at large β (where the Hamiltonian can be approximated as a quadratic potential for $n^2/2$ degrees of freedom).
   - Locate the transition β^* for $n =$ 8,12,16, 20 ($n = 24$ is probably out of reach, but you are welcome to try) by the maximum of $c(\beta)$. An outstanding question is whether the apparent increase of $\beta^*(n)$ with n (for those n that have Hadamard ground states) is just a "finite size effect", and $\beta^*(n)$ approaches a finite limit, or whether $\beta^*(n) \propto n^\phi$ for some positive exponent ϕ. Which hypothesis is best supported by your four data points?
   - How do the entropy changes $\beta^* \Delta e(\beta^*)$ at the transitions depend on n? For this it will probably be easier to look at plots of $\beta(e(\beta) + 1)$ near the transition, rather than the $c(\beta)$ plots you used to locate β^*.
   - What can you say about the model at the oddly-even sizes $n = 10, 14, 18, 22$, which do not have Hadamard ground states?

2. Consider the following very crude model of the low-energy density of states of the $n \times n$ Hadamard model,

$$\rho(E) = \rho_0\, \delta(E - E_0) + \rho_1\, \delta(E - E_1),$$

where the excitation energy $E_1 - E_0 = \Delta = \epsilon\, n^\phi$ is sub-extensive ($\phi < 2$) by the same exponent as the ground state entropy ($\epsilon, \sigma_0, \sigma_1$ are positive constants):

$$\rho_0/\rho_1 = \exp\,(\sigma_0 n^\phi - \sigma_1 n^2).$$

Show that this model has a discontinuity in the mean energy in the limit $n \to \infty$, but at a critical value of $\tilde{\beta}$, where $\beta = \tilde{\beta} n^\psi$, for some exponent ψ. Find ψ and also the behavior of the entropy change, $\beta\Delta$, with n.

3. On clear nights humans are naturally inclined to define the "constellation graph model", or CGM. The vertices of CGM are stars up to a given magnitude (the threshold of vision) and the rule for edges is that each star/vertex is connected to its nearest neighbor. A fundamental quantity in the CGM is the average number of stars S in a connected cluster — a constellation. The stars in CGM are assumed to be distributed uniformly, and their density is high enough that the sky may be modeled as a flat plane (rather than a sphere).

   In class we showed that every constellation has a unique pair of "core stars", defined by the property that each is the other's nearest neighbor, and

$$S = \frac{2}{p_{\mathrm{c}}},$$

   where p_{c} is the probability that a randomly selected star is a core star. Calculate p_{c} in the CGM and thereby determine S.

4. In class we calculated two quantities of central importance in the random graph model:

$$\begin{aligned} f &= \sum_{k=1}^{\infty} k\langle n_k\rangle/n \\ s &= \sum_{k=1}^{\infty} k^2\langle n_k\rangle/n. \end{aligned}$$

We were able to express both of these succinctly in terms of the Lambert w-function defined by

$$w(z) = \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} z^k,$$

for $|z| < 1/e$ and showed that $w(z)$ satisfies the implicit equation

$$z = w(z)e^{-w(z)}.$$

The quantity s is sometimes referred to as the "size" of finite clusters. An alternative definition of finite cluster size is the following:

$$\tilde{s} = \frac{f}{\sum_{k=1}^{\infty}\langle n_k\rangle/n}.$$

This corresponds to the definition we used in the CGM[1].

(a) Contrast in your own *words* the two definitions of "size".

(b) Express $\tilde{s}$ explicitly in terms of the w-function. To get started, first show that the function

$$v(z) = \sum_{k=1}^{\infty} \frac{k^{k-2}}{k!} z^k,$$

satisfies the differential equation $v' = (1-w)w'$ (which is easily integrated).

(c) Using the result of part (b) obtain the explicit formula

$$\tilde{s}(d) = \begin{cases} (1-d/2)^{-1}\,, & d<1 \\ (1-\tilde{d}(d)/2)^{-1}\,, & d>1, \end{cases}$$

where $\tilde{d}(d)$ is the dual-degree function (the non-trivial solution of $\tilde{d}e^{-\tilde{d}} = de^{-d}$). Make a sketch of $\tilde{s}(d)$.

[1] Sadly, there are no percolating mega-constellations in the CGM, so always $f = 1$.