# Assignment 2

Due date: Wednesday, March 6

*Asymptotics of the distribution of field strength*

In lecture we will derive the distribution of electric field strengths $E$ produced by a random distribution of equal point charges in a 3D material. Our solution will be given in terms of the following integral:

$$R(a) = \frac{1}{2\pi^2} \int x \sin x \, e^{-(x/a)^{3/2}} dx,$$

where $a$ is a dimensionless parameter proportional to $E$. Because there is no closed-form expression for this integral we use asymptotic analysis to study its behavior. The first step, in general, is to express the integral in the canonical form

$$I(t) = \int_C f(z) \, e^{t\phi(z)} dz,$$

where $C$ is a suitable contour in the complex plane and we are interested in the limit $t \to \infty$ so that the contour integral is dominated by the contribution at a saddle point or endpoint. We see that $R(a)$ is already of this form if we are interested in the distribution for weak fields, or $a \to 0$. Your assignment is to study the opposite limit, $a \to \infty$.

1. Make the change of variables $(x/a)^{3/2} = z^3$ and express $R(a)$ as the imaginary part of an integral in the canonical form. What are the functions $f$ and $\phi$?
2. Sketch (by hand) level sets of both the real and imaginary parts of $\phi$. The original integration contour $C$ was along the positive real axis with endpoint at the origin. Modify $C$ to take advantage of the saddle point at the origin and thereby obtain the leading behavior of $R(a)$ for large $a$.

*Continued fractions*

In lecture we will derive the formula

$$p(a) = \log_2 \left( 1 + \frac{1}{a(a+2)} \right) \tag{1}$$

for the probability of partial quotients $a = 1, 2, 3, \ldots$ in the continued fraction expansion of a "random" number. Clearly there are exceptions, such as all the quadratic irrationals, whose partial quotients do not have this distribution. What about $\sqrt[3]{2}$? In the first part of this assignments you will design and run a computer program that computes the partial quotients of $\sqrt[3]{2}$ to test whether this number is “random” from the continued fraction perspective. In the second part you will compute the Lypunov exponent of the continued fraction process, a quantity that is directly related to the entropy of the partial quotients.

(a) The *continued fraction process* is the iteration

$$x_{n+1} = \mathrm{cf}(x_n) = \frac{1}{x_n} - \left\lfloor \frac{1}{x_n} \right\rfloor ,$$

where the second (floor operation) term is the partial quotient $a_n$. Here’s a method for computing the partial quotients that only uses integer arithmetic. The trick is to iteratively generate a sequence of polynomials $p_0(x), p_1(x), \ldots$ , all with integer coefficients, and whose only real roots are the iterates $x_0, x_1, \ldots$ in the continued fraction process starting with $x_0 = \sqrt[3]{2}$. For the first polynomial we can use

$$p_0(x) = x^3 - 2,$$

since its only real root is $x_0 = \sqrt[3]{2}$. To generate $p_1(x), p_2(x), \ldots$ we use the reflection and translation transformations

$$\mathrm{R}[p(x)] = x^3\, p\left(\frac{1}{x}\right)$$
$$\mathrm{T}[p(x)] = p(x+1).$$

Verify that these always produce cubic polynomials with integer coefficients and a single real root. Describe how the continued fraction process can be simulated — with only integer arithmetic — by combining these transformations. Explain why the integer coefficients of the polynomials must be able to get arbitrarily large if $x_0$ is indeed “random”. Finally, write a program to generate the first 1000 partial quotients of $\sqrt[3]{2}$. Are the frequencies you find consistent with distribution (1)?

(b) The statement that the partial quotients of a *particular* number are distributed according to (1) rests on the property of ergodicity. Think of the set of possible starting values, $x_0$, as defining an “ensemble” and the index $n$ as a kind of “time”. When ergodicity holds, ensemble averages equal time averages.

Ergodicity is clearly helped when $(\mathrm{cf})^n(x_0) = x_n$ depends sensitively on $x_0$. A standard measure of this sensitivity is the Lyapunov exponent:

$$\lambda = \lim_{n\to\infty} \lim_{\epsilon\to 0} \frac{1}{n} \log \left| \frac{(\mathrm{cf})^n(x_0+\epsilon) - (\mathrm{cf})^n(x_0)}{\epsilon} \right| .$$

By using the chain rule of calculus, calculate $\lambda$ assuming that the iterates $x_0, x_1, \ldots$ are described by the stationary distribution derived in lecture.