Assignment 9

Due date: Friday, November 12

Magnetic force due to moving cylinders of charge

Consider a thin sheet of charge rolled up in the form of a long cylinder of radius R. The sheet has surface charge density σ in its rest frame.

Using Gauss's law, determine the electric field everywhere in the rest frame of the cylinder (neglecting end effects, as usual).

Now suppose you have two identical such cylinders but with opposite sign of charge and occupying the same space so the net charge density is zero. The cylinders are moving with constant speed u along their common axis and have opposite velocity. There is a test charge $q > 0$ that also moves parallel to the cylinder axis; its speed is v and the direction of motion is the same as the positive surface charge. Find the force (magnitude and direction) on the test charge in the frame where it is at rest. Your answer should depend on the charge's distance r from the axis. As in lecture, you need only consider the lowest order effect in the limit of slow speeds.

Some vector calculus facts

The equations of magnetism place even greater demands on our vector calculus skills. Here are three facts for you to prove that we will need in lecture.

1. Let $\mathbf{V}$ be a vector field. Show that

$$\nabla \times (\nabla \times \mathbf{V}) = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}.$$

2. That the electric potential φ of a point charge decays as the inverse distance from the charge is a special instance of the Poisson equation

$$-\nabla^2 \varphi = \rho/\epsilon_0$$

where

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|} \qquad \rho(\mathbf{r}) = q\,\delta^3(\mathbf{r} - \mathbf{r}').$$

($\mathbf{r}'$ is the position of the charge). Use this fact to show that the solution of the Poisson equation with a general source ρ is given by the integral

$$\varphi(\mathbf{r}) = \int d^3\mathbf{r}'\, \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|}.$$

Hint: First apply the Laplacian (with respect to $\mathbf{r}$) to both sides to show that this φ satisfies the Poisson equation. Next argue that if there were two distinct solutions of the Poisson equation for the same source ρ, then the difference of these solutions would give a nonzero solution to the Laplace equation ($\rho = 0$) that vanishes at infinity, which is impossible.

3. Now suppose you have a vector field $\mathbf{V}$ whose divergence is **nonzero** in some finite region of space. Show that it is possible to construct a new vector field $\mathbf{V}'$ from the old one by the addition of the gradient of a scalar function f,

$$\mathbf{V}' = \mathbf{V} + \nabla f,$$

so that $\mathbf{V}'$ has zero divergence everywhere. This $\mathbf{V}'$ will of course have the same curl as $\mathbf{V}$.

Hint: Using part 2 you can write an explicit formula for f.