Assignment 6 Solutions

Electrical resistance between concentric cylinders

Since $\mathbf{E} = -\nabla\varphi$*, we get*

$$\mathbf{E} = -\frac{A}{r}\,\hat{\mathbf{r}}$$

for the given $\varphi(r)$*. This is the electric field of a line charge, which has the same symmetry, and therefore satisfies* $\nabla \cdot \mathbf{E} = 0$ *everywhere (except* $r = 0$*), which covers our case because there is no charge in the region between the two cylinders. Though the constant* A *is still to be determined, from this* $\mathbf{E}$ *we can find the current density:*

$$\mathbf{j} = (1/\rho)\,\mathbf{E} = -\left(\frac{A}{\rho}\right)\frac{1}{r}\,\hat{\mathbf{r}}\ .$$

Now consider a cylindrical surface $\mathcal{S}$ *of radius* r *and length* L *matching the length of the two conductors. The flux of current density through* $\mathcal{S}$ *is the total current* I *flowing between the conductors:*

$$I = \oint_{\mathcal{S}} \mathbf{j} \cdot d\mathbf{a}\ .$$

Since there is no flux through the two ends of $\mathcal{S}$*, we only need the area element* $d\mathbf{a} = |d\mathbf{a}|\hat{\mathbf{r}}$ *on the curved part of the surface,* $\mathcal{S}_c$ *, for the flux integral:*

$$I = -\int_{\mathcal{S}_c}\left(\frac{A}{\rho}\right)\frac{1}{r}\,|d\mathbf{a}| = -\left(\frac{A}{\rho}\right)\frac{1}{r}\int_{\mathcal{S}_c}|d\mathbf{a}| = -\left(\frac{A}{\rho}\right)2\pi L\ .$$

The last step follows from the fact that $\mathcal{S}_c$ *has area* $2\pi r L$*. The difference of potentials gives us the value of* A*,*

$$V = \varphi(a) - \varphi(b) = A\log a - A\log b = A\log(a/b) = -A\log(b/a)\ ,$$

where $A > 0$ *(*$V < 0$*) corresponds to the outer conductor being at the higher potential and positive current flowing inward (*$I < 0$*). Solving for* A *and substituting into the equation for* I *we obtain*

$$I = \left(\frac{V}{\rho\log(b/a)}\right)2\pi L\ ,$$

from which we get the resistance

$$R = V/I = \frac{\rho\log(b/a)}{2\pi L}\ .$$

This is consistent with the general relationship $R = \rho/\mathrm{length}$ *.*

A compound resistor

- Find the ratio of electric fields in the two semiconductors, E_1/E_2, when a potential difference is applied between the ends of the compound resistor.

  Since the same current (net charge per unit time) flows through both conductors,
  $$I_1 = Aj_1 = I_2 = Aj_2$$
  and $j = E/\rho$ we get
  $$E_1/\rho_1 = E_2/\rho_2$$
  or
  $$E_1/E_2 = \rho_1/\rho_2 \ .$$

- Find the ratio of potential differences between the ends of the individual semiconductors, V_1/V_2, when a potential difference is applied between the ends of the compound resistor.

  Since $V = EL$,
  $$\frac{V_1}{V_2} = \frac{E_1 L_1}{E_2 L_2} = \frac{\rho_1 L_1}{\rho_2 L_2} \ ,$$
  where we substituted the electric field ratio from above.

- Find the current I that flows along the compound resistor when the potential difference between its ends is V.

  The total potential change on the compound resistor is
  $$V = V_1 + V_2 = E_1 L_1 + E_2 L_2 \ .$$
  Using the electric field ratio from above we can express this in terms of only E_1:
  $$V = E_1(L_1 + (\rho_2/\rho_1)L_2) \ . \tag{1}$$
  Using conductor 1 to define the common current I,
  $$I = Aj_1 = AE_1/\rho_1 \ ,$$
  we use equation (1) to express E_1 in terms of V and substitute into this equation:
  $$I = A\left(\frac{V}{L_1 + (\rho_2/\rho_1)L_2}\right)\frac{1}{\rho_1} \ .$$
  We see that the resistance of the compound resistor,
  $$R = V/I = \rho_1(L_1/A) + \rho_2(L_2/A) \ ,$$
  is the "series" resistance.

- Explain why in general a surface charge density σ forms where the cylinders are joined and there is a nonzero potential V between the ends of the compound resistor. Calculate the value of σ.

  Just from the fact that there is a different electric field on the two sides of the surface where the conductors make contact, there must be a surface charge density to account for the nonzero net flux of electric field at this surface. Using Gauss's law on a small cylindrical surface that straddles the surface, we obtain

  $$E_2 - E_1 = \sigma/\epsilon_0 \ .$$

  We can again express this in terms of just E_1 using the electric field ratio:

  $$((\rho_2/\rho_1) - 1)E_1 = \sigma/\epsilon_0 \ .$$

  Substituting the E_1 from equation (1) this becomes

  $$((\rho_2/\rho_1) - 1)\left(\frac{V}{L_1 + (\rho_2/\rho_1)L_2}\right) = \sigma/\epsilon_0 \ .$$

  Finally, solving for σ we obtain

  $$\sigma = \epsilon_0 \left(\frac{\rho_2 - \rho_1}{\rho_1 L_1 + \rho_2 L_2}\right) V \ .$$

  The surface charge density can be of either sign, depending on the sign of V (direction of the current) and which of the conductors has the greater resistivity.