# Assignment 5 Solutions

*Easy evaluation of the electric energy density integral*

Prove the identity

$$\nabla \cdot (\mathbf{E}_1 \varphi_2) = (\nabla \cdot \mathbf{E}_1)\varphi_2 + \mathbf{E}_1 \cdot \nabla \varphi_2$$

*Each of the three terms is itself the sum of three terms, one for each axis in three dimensions. Here are just the terms involving $\partial/\partial x$ and $x$-components of vectors:*

$$\frac{\partial}{\partial x}(E_{1x}\varphi_2) = \frac{\partial E_{1x}}{\partial x}\varphi_2 + E_{1x}\frac{\partial \varphi_2}{\partial x} \ .$$

*But this is just the ordinary product rule of calculus. The same identity applies to the $y$ and $z$ counterparts. Summing the three product-rule identities gives the identity you were asked to prove.*

Use the divergence theorem to prove that

$$\int d^3\mathbf{r}\ \nabla \cdot (\mathbf{E}_1 \varphi_2) = 0.$$

*By the divergence theorem,*

$$\int_{\mathcal{V}} d^3\mathbf{r}\ \nabla \cdot (\mathbf{E}_1 \varphi_2) = \oint_{\mathcal{S}} (\mathbf{E}_1 \varphi_2) \cdot d\mathbf{a} \ ,$$

*where $\mathcal{S}$ is the closed surface that bounds the region $\mathcal{V}$. We are free to take $\mathcal{V}$ to be a large sphere of radius $R$ that encloses both point charges with center fixed relative to these charges. In the limit of large $R$ the difference between the sphere center and the positions of the charges is unimportant and $|\mathbf{E}_1| \sim c/R^2$ and $\phi_2 \sim c'/R$, where $c$ and $c'$ are constants, so that the integrand of the surface integral behaves as $1/R^3$ for large $R$. Since the surface area of the sphere only grows as $R^2$, the surface integral decays as $1/R^3 \times R^2 = 1/R$ and is zero in the limit of an infinite sphere (corresponding to the integral over $\mathcal{V}$ being the integral over all space).*

You are now left with evaluating the integral

$$\int d^3\mathbf{r}\ (\nabla \cdot \mathbf{E}_1)\varphi_2.$$

You can evaluate this without doing any serious work. The divergence of the electric field produced by charge 1 evaluates to a simple distribution; express it in terms of the Dirac delta "function".

*The source of the field* $\mathbf{E}_1$ *is a point charge* $q_1$ *located at* $\mathbf{r}_1$*, that is, the charge density*

$$\rho_1(\mathbf{r}) = q_1\,\delta^3(\mathbf{r}-\mathbf{r}_1)\ .$$

*Since* $\nabla\cdot\mathbf{E}_1 = \rho_1/\epsilon_0$ *(local Gauss law), we can rewrite the integral as*

$$\int d^3\mathbf{r}\ (\rho_1(\mathbf{r})/\epsilon_0)\varphi_2(\mathbf{r}) = \int d^3\mathbf{r}\ q_1\,\delta^3(\mathbf{r}-\mathbf{r}_1)\varphi_2(\mathbf{r})/\epsilon_0\ .$$

*Using the delta-function identity this evaluates to*

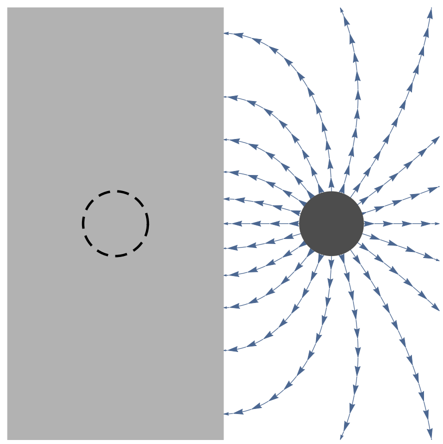
$$(q_1/\epsilon_0)\varphi_2(\mathbf{r}_1) = \frac{K}{\epsilon_0}\frac{q_1 q_2}{|\mathbf{r}_1-\mathbf{r}_2|}\ .$$

Your answer for $U_{12}$ should equal the answer you got in assignment 2 the hard way.

$$\begin{aligned} U_{12} &= \epsilon_0\int d^3\mathbf{r}\ \mathbf{E}_1(\mathbf{r})\cdot\mathbf{E}_2(\mathbf{r}) \\ &= -\epsilon_0\int d^3\mathbf{r}\ \mathbf{E}_1(\mathbf{r})\cdot\nabla\varphi_2(\mathbf{r}) \\ &= \epsilon_0\int d^3\mathbf{r}\ (\nabla\cdot\mathbf{E}_1)\varphi_2 \\ &= K\frac{q_1 q_2}{|\mathbf{r}_1-\mathbf{r}_2|}\ . \end{aligned}$$

*Point charge near the surface of a conductor*

Keeping in mind the rules that apply to field lines near a conductor, make a drawing of the field lines in the $x$-$y$ plane (the 3D pattern is symmetric about the $x$-axis). If this reminds you of a field line drawing you've seen before, state which it was!



*This resembles the field of a dipole in the region $x > 0$. If we had a true dipole, there would also be a charge $-q$ at the center of the dashed circle. By symmetry, the dipole field lines are perpendicular to the plane $x = 0$, exactly as required at the surface of a conductor (the region $x < 0$). In the case of a conductor, instead of a charge $-q$, the field lines terminate at the conductor because there is a negative charge density on its surface.*

Compute the magnitude of this surface field, $E(r)$, as a function of the distance $r$ from the $x$-axis. As a check, your answer should have the form

$$E(r) = \frac{A}{(r^2 + B)^{3/2}},$$

where $A$ and $B$ are constants (to be determined by you).

*The electric field produced by the two point charges, one real and the other fictitious, evaluated at $x = 0$ (technically, just outside the conductor), and arbitrary $y$ is*

$$\begin{aligned} \mathbf{E} &= Kq\left(\frac{-a\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{(a^2+y^2)^{3/2}} - \frac{a\hat{\mathbf{x}} + y\hat{\mathbf{y}}}{(a^2+y^2)^{3/2}}\right) \\ &= Kq\frac{-2a\hat{\mathbf{x}}}{(a^2+y^2)^{3/2}}\ . \end{aligned}$$

*This is perpendicular to the surface of the conductor (the $x = 0$ plane) and has magnitude of the form given in the assignment with*

$$A = 2Kqa, \qquad B = a^2\ .$$

Calculate $\sigma(r)$, using Gauss's law, as we did in lecture. Integrate this surface charge density over the entire plane $x = 0$ to find out how much total charge resides on the surface of the conductor.

*As shown in lecture, the electric field magnitude at the surface of a conductor and the surface charge density there are related as $E = \sigma/\epsilon_0$. The sign of $\sigma$ is positive when the field lines point away from the conductor, so in this case $\sigma$ is negative and has value*

$$\sigma(r) = -\epsilon_0 E(r) = -\frac{q}{2\pi}\frac{a}{(a^2 + r^2)^{3/2}}$$

*since $\epsilon_0 K = 1/(4\pi)$. Integrating this density over the entire conductor surface,*

$$\int_0^\infty (2\pi r dr)\sigma(r) = -q \ ,$$

*we see that the system (point charge $+q$ and surface charge) is neutral.*

Calculate the electric force, magnitude and direction, experienced by the $+q$ charge using whichever source (real surface charge or fictitious point charge) is simpler for the task.

*From the local perspective it is the electric field $\mathbf{E}$ at the location of the $+q$ charge that is responsible for the force $\mathbf{F} = q\mathbf{E}$ on the charge. We know that $\mathbf{E}$ is produced by the surface charges, but exactly the same $\mathbf{E}$ (and force) would be produced by a $-q$ charge at $x = -a$ and no surface charge:*

$$\mathbf{F} = q\left(K\frac{(-q)}{(2a)^2}\,\hat{\mathbf{x}}\right) = -K\frac{q^2}{4a^2}\,\hat{\mathbf{x}} \ .$$