Assignment 5

Due date: Friday, October 1

Easy evaluation of the electric energy density integral

In the second assignment you evaluated the integral of the electric energy density in space produced by a pair of point charges. The calculation was quite involved, with complications arising from working with vector fields in a cylindrical coordinate system. This problem shows how the same integral can be evaluated much more simply by working with the scalar electric potential function $\varphi(\mathbf{r})$. It also reviews the divergence theorem and the Dirac delta function.

The integral to be evaluated is

$$U_{12} = \epsilon_0 \int d^3\mathbf{r}\ \mathbf{E}_1(\mathbf{r}) \cdot \mathbf{E}_2(\mathbf{r}),$$

where $\mathbf{E}_1$ is the field of a point charge q_1 at $\mathbf{r}_1$ and $\mathbf{E}_2$ is the field of a point charge q_2 at $\mathbf{r}_2$. We rewrite this in terms of the electric potential function associated with $\mathbf{E}_2$:

$$U_{12} = -\epsilon_0 \int d^3\mathbf{r}\ \mathbf{E}_1(\mathbf{r}) \cdot \nabla\varphi_2(\mathbf{r})$$

$$\varphi_2(\mathbf{r}) = \frac{K q_2}{|\mathbf{r} - \mathbf{r}_2|}.$$

We made the usual choice for the arbitrary constant in the definition of φ_2, such that this function vanishes at infinity.

Prove the identity

$$\nabla \cdot (\mathbf{E}_1 \varphi_2) = (\nabla \cdot \mathbf{E}_1)\varphi_2 + \mathbf{E}_1 \cdot \nabla\varphi_2$$

which applies in fact to the product of any vector field and scalar function (i.e. its validity does not depend on specific properties of the electric field and potential). *Hint:* Examine one component of the gradient at a time, say $\partial/\partial x$, in all three terms and observe that this is just the ordinary product rule of calculus.

After proving the identity replace the integrand in U_{12}, one term in the identity, with the other two terms.

Use the divergence theorem to prove that

$$\int d^3\mathbf{r}\ \nabla\cdot(\mathbf{E}_1\varphi_2) = 0.$$

Hint: How does the vector field quantity in parentheses behave as you approach infinity? Without doing any integral, can you argue the flux is zero through the "surface at infinity"?

You are now left with evaluating the integral

$$\int d^3\mathbf{r}\ (\nabla\cdot\mathbf{E}_1)\varphi_2.$$

You can evaluate this without doing any serious work. The divergence of the electric field produced by charge 1 evaluates to a simple distribution; express it in terms of the Dirac delta "function".

The final step makes use of the following identity (f is an arbitrary scalar function):

$$\int d^3\mathbf{r}\ \delta^3(\mathbf{r}-\mathbf{r}_1)f(\mathbf{r}) = f(\mathbf{r}_1).$$

You can understand this identity as follows (there is actually nothing to prove): The function $\delta^3(\mathbf{r}-\mathbf{r}_1)$ is defined so it has unit integral; all that is different here is that it is being multiplied by a scalar function $f(\mathbf{r})$. If this function were a constant you would already know what to do. On the other hand, since the Dirac function is zero except when $\mathbf{r}$ lies in a small neighborhood of $\mathbf{r}_1$, you may as well replace $f(\mathbf{r})$ by its value at $\mathbf{r}=\mathbf{r}_1$ (thereby replacing $f(\mathbf{r})$ by the constant $f(\mathbf{r}_1)$).

Use the above identity to evaluate the final integral. Your answer for U_{12} should equal the answer you got in assignment 2 the hard way.

Point charge near the surface of a conductor

In this problem you will study the electric field produced by a point charge when it is placed near the surface of a conductor, the charge density that develops on the surface of the conductor, and the force experienced by the point charge as a result of this surface charge. We will keep the geometry as simple as possible. The conductor is the entire region $x<0$; its surface is the plane $x=0$. The point charge $q>0$ is placed on the positive x-axis at $x=a$.

Keeping in mind the rules that apply to field lines near a conductor, make a drawing of the field lines in the x-y plane (the 3D pattern is symmetric about the x-axis). If this reminds you of a field line drawing you've seen before, state which it was!

An electric field that has all the required properties in the region $x>0$ can be constructed by placing a "fictitious" charge $-q$ on the x-axis at $x=-a$, and letting it together with the actual $+q$ charge at $x = +a$ be sources of the field (we ignore what is going on in the region $x < 0$ for now). Using symmetry, show that the net electric field right at the surface of the conductor, $x = 0$, is properly perpendicular to the surface. Compute the magnitude of this surface field, $E(r)$, as a function of the distance r from the x-axis. As a check, your answer should have the form

$$E(r) = \frac{A}{(r^2 + B)^{3/2}},$$

where A and B are constants (to be determined by you).

Now suppose the electric field really *is* zero inside the conductor, $x < 0$. In order for it to abruptly jump to the non-zero value you found above at $x = 0$, there must be some surface charge density $\sigma(r)$. Calculate $\sigma(r)$, using Gauss's law, as we did in lecture. Integrate this surface charge density over the entire plane $x = 0$ to find out how much total charge resides on the surface of the conductor.

By introducing the surface charge the electric field is able to jump from a nonzero value just outside the conductor to zero inside the conductor. Consequently, the $+q$ charge together with the surface charge account for the entire electric field, outside *and* inside the conductor; there is no need for a fictitious $-q$ charge inside the conductor. The role of the fictitious charge was simply to help us work out the surface charge. On the other hand, if we are mostly interested in the electric field in the region outside the conductor, we can think of the fictitious charge as an alternative, but equivalent, source. Calculate the electric force, magnitude and direction, experienced by the $+q$ charge using whichever source (real surface charge or fictitious point charge) is simpler for the task.