Assignment 2

Due date: Friday, September 10

Electric energy density

The electric field not only confers locality to the nature of the electric force, but locality to the origin of **electric energy**.

Starting from Coulomb's force law and the work-energy theorem for conservative forces, we obtained in lecture a formula for the energy of a collection of point charges. This formula is non-local, *i.e.* it is not the sum of contributions of locally defined quantities.

In this assignment you will learn how to compute the energy of a collection of point charges just from the electric field they produce. The key idea is to construct a scalar energy density from the electric field, $\mathbf{E}(\mathbf{r})$. Since the electric field is a vector, one of the simplest local scalars we can construct is its dot product:

$$u(\mathbf{r}) = K' \, \mathbf{E}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) = K' \, |\mathbf{E}(\mathbf{r})|^2 \, .$$

This is the **electric energy density**; the constant K' is at this stage unknown. What are the units of K'?

Divergent energy

Your task is to show that this energy density, when integrated over all of space, correctly reproduces the energy of a system of point charges which you already know. First consider a system of two charges. Charge 1 produces an electric field $\mathbf{E}(\mathbf{r}; \mathbf{r}_1, q_1)$, where $\mathbf{r}_1$ is its position and q_1 the value of its charge. Similarly, charge 2 produces the field $\mathbf{E}(\mathbf{r}; \mathbf{r}_2, q_2)$. One of the first things you will have to confront is the fact that the electric energy density of a point charge rises very steeply near the charge and its integral over space actually diverges.

Explain, in quantitative terms, why the integral of $u(\mathbf{r})$ diverges for the field of a single point charge.

Subtracting an infinite constant

The divergence of the electric field energy produced by point charges does not stand in your way to **compare** field energies produced by systems of point charges when their relative positions are changed. Say we have point charges at positions $\mathbf{r}_1$ and $\mathbf{r}_2$. We define the energy of the system like this:

$$U_{12} = K' \int d^3\mathbf{r} \; |\mathbf{E}(\mathbf{r}; \mathbf{r}_1, q_1) + \mathbf{E}(\mathbf{r}; \mathbf{r}_2, q_2)|^2 - K' \int d^3\mathbf{r} \; |\mathbf{E}(\mathbf{r}; 0, q_1) + \mathbf{E}(\mathbf{r}; \infty, q_2)|^2 \, .$$

We have subtracted the integral of the energy density for the case when the two charges are infinitely far apart (one at the origin, the other at ∞), as a convenient point of reference.

Show that

$$U_{12} = K' \int d^3\mathbf{r}\; 2\,\mathbf{E}(\mathbf{r};\mathbf{r}_1,q_1)\cdot\mathbf{E}(\mathbf{r};\mathbf{r}_2,q_2).$$

The energy integrand

Place charge 1 at the origin and charge 2 at distance R on the positive z-axis. This is the configuration for which you will compute U_{12}.

Sketch or have a computer plot the integrand of the U_{12} integral in a plane that passes through the z-axis, say for a pair of like charges. Where does it change sign? Though the *integrand* still diverges, argue that the integral U_{12} is now finite.

The double integral

Set up the U_{12} integral in cylindrical coordinates and evaluate the angular part of the integral (for which the integrand is constant). You will end up with a double integral over z and the distance r from the z-axis.

Computing the double integral

Perform the r-integral using the substitution $t = r^2$ and the formula

$$\int_0^\infty \frac{dt}{2}\frac{A+t}{[(B+t)(C+t)]^{3/2}} = \frac{1+A/\sqrt{BC}}{B+C+2\sqrt{BC}}.$$

You now have left an integral over z. Examine the integrand in each of the three regions: $z<0$, $0<z<R$, and $R<z$; it should greatly simplify in each region.

Compare with the non-local formula

Confirm that your result for U_{12} agrees with the non-local, Coulomb-law derived formula from lecture when the constant K' has a particular value.

Multiple point charges

Show that integrating the appropriately subtracted electric energy density for multiple point charges always reduces to the case of pairs that you have worked out (and therefore agrees with the energy formula given in lecture).