Assignment 2 Solutions

Electric energy density

The electric field not only confers locality to the nature of the electric force, but locality to the origin of **electric energy**.

Starting from Coulomb's force law and the work-energy theorem for conservative forces, we obtained in lecture a formula for the energy of a collection of point charges. This formula is non-local, *i.e.* it is not the sum of contributions of locally defined quantities.

In this assignment you will learn how to compute the energy of a collection of point charges just from the electric field they produce. The key idea is to construct a scalar energy density from the electric field, $\mathbf{E}(\mathbf{r})$. Since the electric field is a vector, one of the simplest local scalars we can construct is its dot product:

$$u(\mathbf{r}) = K' \, \mathbf{E}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) = K' \, |\mathbf{E}(\mathbf{r})|^2 .$$

This is the **electric energy density**; the constant K' is at this stage unknown. What are the units of K'?

Since (just units) $E = KQ/L^2$ (Q = charge, L = length), and $U = KQ^2/L$ is energy, we see that $E^2 = KU/L^3$ (again, just units), so K' should have units of $1/K$ in order for $K'E^2$ to be an energy density (energy per volume).

Divergent energy

Your task is to show that this energy density, when integrated over all of space, correctly reproduces the energy of a system of point charges which you already know. First consider a system of two charges. Charge 1 produces an electric field $\mathbf{E}(\mathbf{r}; \mathbf{r}_1, q_1)$, where $\mathbf{r}_1$ is its position and q_1 the value of its charge. Similarly, charge 2 produces the field $\mathbf{E}(\mathbf{r}; \mathbf{r}_2, q_2)$. One of the first things you will have to confront is the fact that the electric energy density of a point charge rises very steeply near the charge and its integral over space actually diverges.

Explain, in quantitative terms, why the integral of $u(\mathbf{r})$ diverges for the field of a single point charge.

The volume element in spherical coordinates, for a spherically symmetric integrand after integrating over angles, is $4\pi r^2 dr$. Integrating between a distance $r = r_{\min}$ from the charge and $r = \infty$, we get

$$U = \int_{r_{\min}}^{\infty} K' \frac{(Kq)^2}{r^4} 4\pi r^2 dr = 4\pi K' K^2 q^2 \frac{1}{r_{\min}},$$

for the energy (stored in the electric field) for a charge q. This diverges as $r_{\min} \to 0$.

Subtracting an infinite constant

The divergence of the electric field energy produced by point charges does not stand in your way to **compare** field energies produced by systems of point charges when their relative positions are changed. Say we have point charges at positions $\mathbf{r}_1$ and $\mathbf{r}_2$. We define the energy of the system like this:

$$U_{12} = K' \int d^3\mathbf{r}\ \left|\mathbf{E}(\mathbf{r};\mathbf{r}_1,q_1)+\mathbf{E}(\mathbf{r};\mathbf{r}_2,q_2)\right|^2 - K' \int d^3\mathbf{r}\ \left|\mathbf{E}(\mathbf{r};0,q_1)+\mathbf{E}(\mathbf{r};\infty,q_2)\right|^2 .$$

We have subtracted the integral of the energy density for the case when the two charges are infinitely far apart (one at the origin, the other at ∞), as a convenient point of reference.

Show that

$$U_{12} = K' \int d^3\mathbf{r}\ 2\,\mathbf{E}(\mathbf{r};\mathbf{r}_1,q_1)\cdot\mathbf{E}(\mathbf{r};\mathbf{r}_2,q_2).$$

Expanding the two $|\cdots|^2$ we get six terms. Two of these have the form

$$K' \int d^3\mathbf{r}\ \left|\mathbf{E}(\mathbf{r};\mathbf{r}_{\rm source},q_1)\right|^2$$

differing only in the position of the source and a sign. They are therefore equal in magnitude (identical instances of the divergent integral above) and cancel by virtue of the sign. The same holds for the two terms involving only q_2. We are left with two terms proportional to $q_1 q_2$. The term

$$K' \int d^3\mathbf{r}\ 2\,\mathbf{E}(\mathbf{r};0,q_1)\cdot\mathbf{E}(\mathbf{r};\infty,q_2)$$

is zero because there is no $\mathbf{r}$ *(integration point) that is not infinitely far from at least one of the charges (for which the electric field is zero). Thus only the U_{12} cross-term remains.*

The energy integrand

Place charge 1 at the origin and charge 2 at distance R on the positive z-axis. This is the configuration for which you will compute U_{12}.

Sketch or have a computer plot the integrand of the U_{12} integral in a plane that passes through the z-axis, say for a pair of like charges. Where does it change sign? Though the *integrand* still diverges, argue that the integral U_{12} is now finite.

Using

$$\begin{aligned}\mathbf{r}-\mathbf{r}_1 &= x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+z\,\hat{\mathbf{z}}\\ \mathbf{r}-\mathbf{r}_2 &= x\,\hat{\mathbf{x}}+y\,\hat{\mathbf{y}}+(z-R)\,\hat{\mathbf{z}}\end{aligned}$$

we obtain[1]

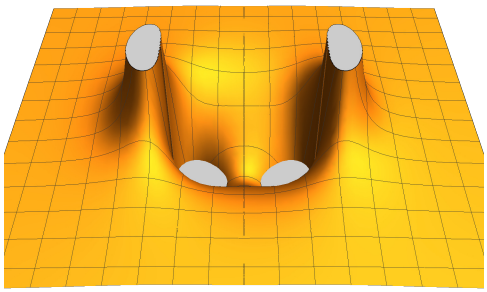
$$U_{12} = 2K'K^2 q_1 q_2 \int d^3\mathbf{r} f(\mathbf{r})$$

$$f(\mathbf{r}) = \frac{\rho^2 + z(z-R)}{(\rho^2+z^2)^{3/2}(\rho^2+(z-R)^2)^{3/2}}$$

where $\rho = \sqrt{x^2+y^2}$ *is the distance to the* z*-axis. The function* $f(\mathbf{r})$ *diverges at two places on the* z*-axis (*$\rho = 0$*), at* $z = 0$ *and* $z = R$ *(the positions of the two charges). However, the divergence is different from what you saw earlier, for* $|E|^2$*, because of the numerator. Rewriting this as*

$$\left(x^2 + y^2 + (z - R/2)^2\right) - (R/2)^2,$$

we see that the numerator is negative when the distance to the point $(R/2)\hat{\mathbf{z}}$ *is less than* $R/2$ *and positive otherwise. In other words,* $f(\mathbf{r})$ *changes sign everywhere on the surface of a sphere of radius* $R/2$ *centered at* $(R/2)\hat{\mathbf{z}}$*. This sphere passes through both charges. As a result, "half" of the diverging behavior is positive and the other half is negative at each of the charges. But it is not just "cancellation by sign" that makes the integral finite. Consider the part of the integral near the charge at* $\mathbf{r} = 0$*. The spherical volume element centered on this charge, ignoring angles, goes as* $r^2 dr$*, while the electric field magnitude from that charge goes as* $1/r^2$*. The electric field from the other charge can be approximated as a constant in a small neighbotrhood of* $\mathbf{r} = 0$*. These facts combined show that the* r*-dependence of the integrand near* $\mathbf{r} = 0$ *is* r^2/r^2 *and does not diverge.*



[1] We are using ρ instead of r for the distance from the z-axis and reserve r for the distance to the origin in spherical coordinates.

The double integral

Set up the U_{12} integral in cylindrical coordinates and evaluate the angular part of the integral (for which the integrand is constant). You will end up with a double integral over z and the distance ρ from the z-axis.

Since $f(\mathbf{r})$ does not depend on the angle ϕ about the cylinder axis, the ϕ-integral just gives a factor of 2π. Combining constants as

$$C = 2 \times 2\pi K' K^2 q_1 q_2,$$

we arrive at

$$U_{12} = C \int_{-\infty}^{\infty} dz \int_0^{\infty} \rho d\rho \, \frac{\rho^2 + z(z-R)}{(\rho^2 + z^2)^{3/2} (\rho^2 + (z-R)^2)^{3/2}} \, .$$

Computing the double integral

Perform the ρ-integral using the substitution $t = \rho^2$ and the formula

$$\int_0^{\infty} \frac{dt}{2} \frac{A + t}{[(B+t)(C+t)]^{3/2}} = \frac{1 + A/\sqrt{BC}}{B + C + 2\sqrt{BC}}.$$

You now have left an integral over z. Examine the integrand in each of the three regions: $z < 0$, $0 < z < R$, and $R < z$; it should greatly simplify in each region.

Doing the ρ integral first with the substitution $t = \rho^2$, $dt = 2\rho d\rho$, we get an integral of the form above with positive constants B and C. Remembering that $\sqrt{z^2} = |z|$ and $\sqrt{(z-R)^2} = |z - R|$, the ρ-integral evaluates to

$$\frac{1 + z(z-R)/|z(z-R)|}{z^2 + (z-R)^2 + 2|z(z-R)|} \, .$$

When z satisfies $0 < z < R$, $|z(z-R)| = -z(z-R)$ and we see that the numerator becomes $1 - 1 = 0$. In both of the other regions, $z < 0$ and $z > R$, we instead have $|z(z-R)| = z(z-R)$ and the expression above simplifies to

$$\frac{2}{(2z-R)^2} \, .$$

Finally, doing the z-integral we obtain

$$\int_{-\infty}^{0} \frac{2dz}{(2z-R)^2} + \int_R^{\infty} \frac{2dz}{(2z-R)^2} = \frac{1}{R} + \frac{1}{R} = \frac{2}{R} \, .$$

Combining this with the constant C defined earlier, and remembering that $R = r_{12}$ is the distance between the charges,

$$U_{12} = 8\pi K' K^2 \, \frac{q_1 q_2}{r_{12}} \, .$$

Compare with the non-local formula

Confirm that your result for U_{12} agrees with the non-local, Coulomb-law derived formula from lecture when the constant K' has a particular value.

Our energy density integral for U_{12} agrees with the work-integral from lecture with the parameter value

$$K' = \frac{1}{8\pi K} \, .$$

Multiple point charges

Show that integrating the appropriately subtracted electric energy density for multiple point charges always reduces to the case of pairs that you have worked out (and therefore agrees with the energy formula given in lecture).

With multiple charges there will be pairs of integrals of $+|\mathbf{E}|^2$ and $-|\mathbf{E}|^2$ with source at each of the point charges (these cancel), and integrals of $2\mathbf{E}(\mathbf{r}; q_i, \mathbf{r}_i) \cdot \mathbf{E}(\mathbf{r}; q_j, \mathbf{r}_j)$ for all pairs of charges i and j — when located at their actual positions $\mathbf{r}_i$ and $\mathbf{r}_j$ — as well as the same expression but with $\mathbf{r}_i$ and $\mathbf{r}_j$ replaced by positions infinitely far apart. The latter pair-terms vanish while the former give the correct pair-energies when integrated, as we worked out earlier for the case of just two charges.