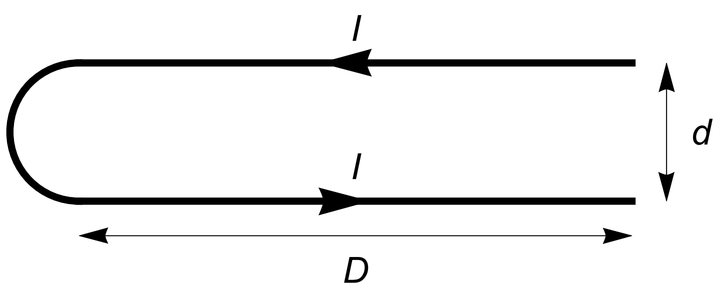
Assignment 12

Due date: Wednesday, December 3

Parallel-sheet inductor

An inductor is comprised of two parallel conducting sheets joined together (by a half-cylinder) so current I can flow in opposite directions on the two sheets as shown below the 3D rendering:



The current I is distributed uniformly across the width w of the sheet (the dimension into the page in the 2D diagram). Your task is to determine the inductance L of this device in the limit where the spacing d is much smaller than both D and w, so that end-effects can be ignored. In particular, the magnetic field $\mathbf{B}$ in this limit will be very uniform between the sheets — like the electric field between plates of a parallel plate capacitor — and the field at any of the four edges of the “sheet sandwich” can be neglected in your calculations.

1. What is the direction of the uniform $\mathbf{B}$ between the plates? How did you arrive at your answer?
2. Assuming that $\mathbf{B} = 0$ above and below the sheets, use the integral form of Ampère’s law to obtain the magnitude of $\mathbf{B}$ between the sheets.

3. In the EMF integral for the inductor,

$$\mathcal{E} = \oint_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{r},$$

use a closed curve $\mathcal{C}$ that includes the entire path taken by current flowing between the ends of the inductor. The same curve bounds the surface $\mathcal{S}$ through which the flux of magnetic field should be calculated in Faraday's law. Calculate the magnetic flux Φ_B through $\mathcal{S}$.

4. Since by Faraday's law (up to signs) $\mathcal{E} = d\Phi_B/dt = L\, dI/dt$, you can now calculate the inductance L. Check that your answer has the form $\mu_0 \times$ length.

5. As another check, compare the device energy for this inductor, $\frac{1}{2}LI^2$, with the energy you get by integrating the uniform magnetic energy density $B^2/(2\mu_0)$ over the volume between the sheets.

The moving slab of electromagnetic field

Perhaps even simpler than the sinusoidal wave described in lecture is the "moving slab", or electromagnetic pulse. This is a propagating configuration of electric and magnetic fields, where the dynamics is purely the result of moving boundaries. The boundaries are two infinite planes: $z = a$ and $z = b$, where $a < b$ and both a and b are increasing with time. The electric and magnetic fields are nonzero only within the slab, i.e. the region $a < z < b$, where they are uniform: $\mathbf{E} = E\,\hat{\mathbf{x}}$, $\mathbf{B} = B\,\hat{\mathbf{y}}$. In this exercise you will confirm, that for a particular relationship between the constant values E and B, and just the right boundary velocities $\dot{a}$ and $\dot{b}$, all the Maxwell equations are satisfied.

Check that the divergence equations are satisfied. This breaks down into two parts for each of the fields. Checking that $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$ within the slab is very easy. However, you also have to check that there are no problems at the boundaries. To do that, use the integrated laws and consider small cubic boxes that are fixed in space and straddle the boundary (fields on one half, vacuum on the other) and check that the flux into them is zero.

Check that the curl equations are satisfied. Again, this is very easy inside the slab. To check the laws at the boundaries you have to use the integrated forms with rectangular loops that straddle them and are fixed in space. By orienting the loops appropriately, you will be able to relate the line integral of $\mathbf{E}$ to the changing flux of $\mathbf{B}$, and vice versa. Do this at both boundaries. You will see that to satisfy both equations (at each boundary) the velocities $\dot{a}$ and $\dot{b}$ have to have a particular value and E and B need to be in a particular ratio.

Is the sign of the slab velocity consistent with the Poynting vector, $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$?