Assignment 11 Solutions

Lorentz-invariance of Ampere's law

$$\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

velocity of primed frame relative to unprimed frame:

$$\vec{V} = v\hat{x}$$

field transformation:

$$E_x' = E_x \quad E_y' = \gamma(E_y - vB_z) \quad E_z' = \gamma(E_z + vB$$

$$B_x' = B_x \quad B_y' = \gamma(B_y + \frac{v}{c^2}E_z) \quad B_z' = \gamma(B_z - \frac{v}{c^2}E_y$$

derivative transformation:

$$\frac{\partial}{\partial t'} = \gamma\frac{\partial}{\partial t} + \gamma v\frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial x'} = \gamma\frac{\partial}{\partial x} + \gamma\frac{v}{c^2}\frac{\partial}{\partial t}$$

$$\frac{\partial}{\partial y'} = \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$$

charge-current density trans.:

$$\rho' = \gamma\left(\rho - \frac{v}{c^2} j_x\right)$$

$$j_x' = \gamma\left(j_x - v\rho\right)$$

$$j_y' = j_y$$

$$j_z' = j_z$$

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x-component:

$$\frac{\partial B_z'}{\partial y'} - \frac{\partial B_y'}{\partial z'} - \frac{1}{c^2}\frac{\partial E_x'}{\partial t'} \overset{?}{=} \mu_0 j_x'$$

$$\gamma\left(\frac{\partial B_z}{\partial y} - \frac{v}{c^2}\frac{\partial E_y}{\partial y}\right) - \gamma\left(\frac{\partial B_y}{\partial z} + \frac{v}{c^2}\frac{\partial E_z}{\partial z}\right)$$

$$-\frac{1}{c^2}\left(\gamma\frac{\partial}{\partial t} + \gamma v\frac{\partial}{\partial x}\right)E_x \overset{?}{=} \mu_0\gamma\left(j_x - v\rho\right)$$

$$\underbrace{\gamma\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{1}{c^2}\frac{\partial E_x}{\partial t}\right)}_{\mu_0 j_x \checkmark} \underbrace{- \gamma\frac{v}{c^2}\underbrace{\vec{\nabla}\cdot\vec{E}}_{\rho/\varepsilon_0}}_{-\gamma v \mu_0 \rho \checkmark} \overset{!}{=} \mu_0\gamma j_x - \mu_0\gamma v\rho$$

y-component:

$$\frac{\partial B'_x}{\partial z'} - \frac{\partial B'_z}{\partial x'} - \frac{1}{c^2}\frac{\partial E'_y}{\partial t'} \overset{?}{=} \mu_0 j'_y$$

$$\frac{\partial B_x}{\partial z} - \gamma^2\left(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}\right)\left(B_z - \frac{v}{c^2}E_y\right)$$

$$- \frac{1}{c^2}\gamma^2\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)\left(E_y - vB_z\right) \overset{?}{=} \mu_0 j$$

Terms marked ① and ② cancel.

$$\frac{\partial B_x}{\partial z} - \underbrace{\gamma^2\left(1 - \frac{v^2}{c^2}\right)}_{1}\frac{\partial B_z}{\partial x} - \underbrace{\frac{\gamma^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)}_{1/c^2}\frac{\partial E_y}{\partial t}$$

$$\overset{!}{=} \mu_0 j_y$$

Since the z-component is also perpendicular to $\vec{v}$, it works out in the same way as the y-compone

Gauss law:

$$\vec{\nabla}'\cdot\vec{E}' \stackrel{?}{=} \rho'/\epsilon_0$$

$$\gamma\left(\frac{\partial}{\partial x}+\frac{v}{c^2}\frac{\partial}{\partial t}\right)E_x + \frac{\partial}{\partial y}\gamma\left(E_y - vB_z\right)$$

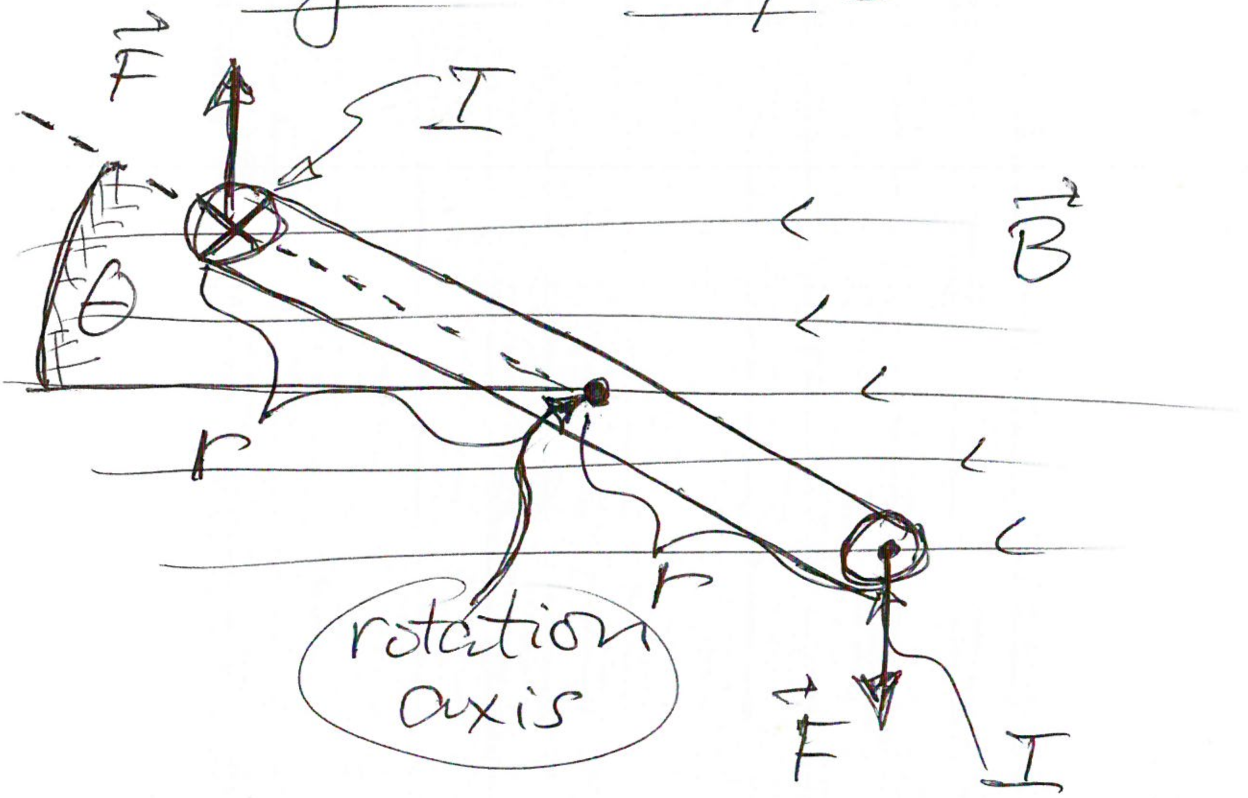
$$+ \frac{\partial}{\partial z}\gamma\left(E_z + vB_y\right) \stackrel{?}{=} \frac{1}{\epsilon_0}\gamma\left(\rho - \frac{v}{c^2}j_x\right)$$

$$\gamma\underbrace{\vec{\nabla}\cdot\vec{E}}_{\rho/\epsilon_0} - \gamma v\underbrace{\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \frac{1}{c^2}\frac{\partial E_x}{\partial t}\right)}_{\mu_0 j_x = \frac{1}{\epsilon_0 c^2} j_x}$$

$$\stackrel{!}{=} \gamma\frac{\rho}{\epsilon_0} - \gamma v\frac{1}{\epsilon_0 c^2}j_x$$

energy balance with magnetic torque

1.



Only $\vec{F} = IL\hat{L} \times \vec{B}$ on two sides of length L is shown.

$d\vec{F}$ on other sides is always parallel to rotation axis, so does not produce torque about that axis

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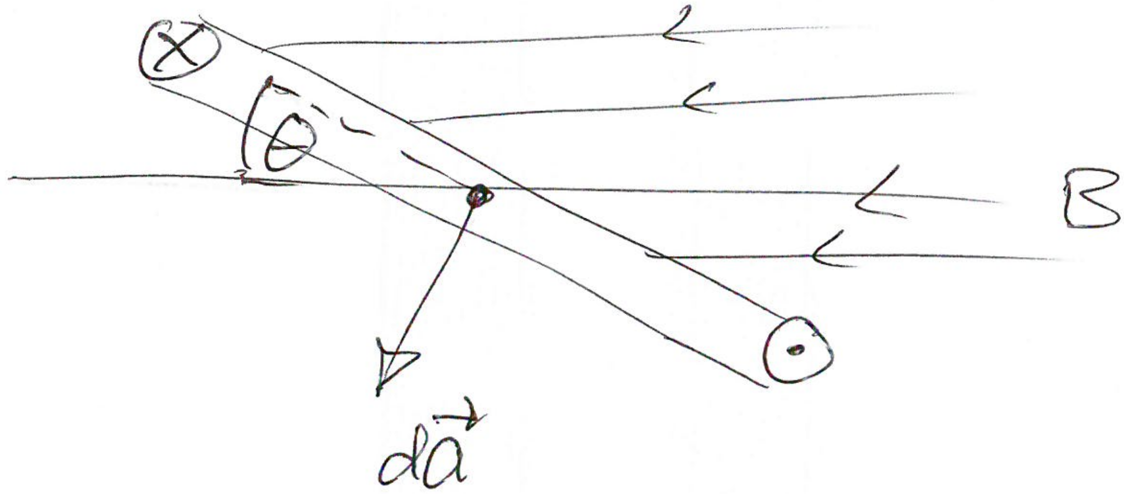
2. $\tau = 2 \times F \cdot \underbrace{r\cos\theta}_{\text{"lever arm"}}$

$= 2IBLr\cos\theta$

$P_{rot} = \omega\tau = 2\omega IBLr\cos\theta$

3.



$\vec{B} \cdot d\vec{a} = B|d\vec{a}|\underbrace{\cos(\pi/2 - \theta)}_{\sin\theta}$

$\Phi_B(\theta) = (B\sin\theta) \cdot (2rL)$

$\mathcal{E}_F = -\dot{\Phi}_B = -(B\cos\theta)\omega(2rL)$

Current in the wire loop would lose energy at rate $I\mathcal{E}_F$, if the battery did not perform extra work to make up for this loss. Notice that

$$-I\mathcal{E}_F = 2\omega I B L r \cos\theta$$

$$= \omega\tau = P_{rot},$$

so the extra work matches exactly the work that goes into increasing the rotational mechanical energy.