Assignment 11

Due date: Friday, December 3

Lorentz invariance of the Ampère and electric-Gauss laws

In lecture 36 we checked the Lorentz invariance of the two Maxwell equations without sources, Faraday and magnetic-Gauss. Now it's your turn to do the same for the equations with sources, Ampère and electric-Gauss:

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0\,\mathbf{j}\,, \qquad \nabla \cdot \mathbf{E} = \rho/\epsilon_0\,.$$

As in lecture, you may boost in the x-direction without sacrificing generality. The transformation of the $(\rho, \mathbf{j})$ 4-vector can be found in the lecture 37 notes.

Energy balance with magnetic torque

At the end of lecture 32, and also in the magnetic dipole problem of homework assignment 10, we saw that a magnetic field exerts a torque on current flowing in a loop and should therefore impart mechanical rotational energy to the wire etc. that is carrying the current. But how is an increase of mechanical energy possible when, fundamentally, the magnetic force never performs work on a charge because it always acts perpendicular to the velocity of the charge? The resolution of this puzzle is that the source of energy is not work performed by the magnetic field, but the mechanism (battery, etc.) that is driving the current. This problem will help you flesh out the details.

The relationship you will prove is

$$P_{\text{emf}} = P_{\text{heat}} + P_{\text{rot}},$$

where two of the terms you already know from the study of circuits. A battery with electromotive force (chemical energy per unit charge) $\mathcal{E}$ provides power $P_{\text{emf}} = \mathcal{E}I$, when the current flowing through it is I. Normally — in the absence of a magnetic torque — this equals the power dissipated as heat in the resistance R of the wire that carries the current, $P_{\text{heat}} = I^2R$. To understand the third power term we will need Faraday's law.

1. Start by making a sketch of the wire loop, magnetic field $\mathbf{B}$, and magnetic forces acting on the sides of the loop. Since 3D drawings are tricky, make a 2D view showing a rectangular loop whose plane is tipped by angle θ relative to

the magnetic field. The loop's dimensions are $L \times (2R)$, and the loop's axis of rotation bisects the edges of length $2R$ (so the edges of length L keep a distance R from the axis as the loop rotates). The direction of the current I gives the loop C an orientation and defines the normal direction to its spanning surface S by the right-hand-rule. In your drawing choose the direction of the current so the normal to S would point in the same direction as $\mathbf{B}$ when $\theta = \pi/2$. However, your drawing should be for a general angle, not that special case.

Recall that the formula for magnetic force is $I\ell\hat{\ell} \times \mathbf{B}$ for a wire of length ℓ with current flowing in the direction $\hat{\ell}$. Use this to attach magnetic force vectors to the two sides of length L. Argue that the forces on the other sides do not contribute to a torque about the axis of rotation defined by θ. Draw the loop at an angle where the torque is acting to *increase* θ.

2. When $\dot{\theta} = \omega$ is positive and the torque $\tau > 0$ is acting to increase θ, then the rate of rotational energy change (power) $P_{\text{rot}} = \omega\tau$ is also positive. Calculate P_{rot} in terms of ω, θ, L, R, and $B = |\mathbf{B}|$.

3. Calculate the flux of magnetic field through the surface $S(\theta)$ spanned by the loop at angle θ using the conventions above for the direction of the surface normal:
$$\Phi_B(\theta) = \int_{S(\theta)} \mathbf{B} \cdot d\mathbf{a} \ .$$
Faraday's law relates the time rate of change of Φ_B to the integral of the electric field around C:
$$\mathcal{E}_F = \oint_C \mathbf{E} \cdot d\mathbf{r} = -\frac{d\Phi_B}{dt}.$$
Calculate $\mathcal{E}_F$ using the fact that the time-dependence of Φ_B is via its dependence on θ and $\dot{\theta} = \omega$. By our sign conventions your $\mathcal{E}_F$ should be negative.

4. Current I flowing around C experiences three energetic processes: (1) gain at rate $I\mathcal{E}$ from the battery, (2) the power $I\mathcal{E}_F$ due to the electric field of Faraday's law, and (3) the loss at rate I^2R to heat:
$$I\mathcal{E} + I\mathcal{E}_F = I^2R.$$
For the θ's you are considering, where P_{rot} is positive, the term $I\mathcal{E}_F$ is negative and the battery term would have to *provide additional power* $-I\mathcal{E}_F$ in order to keep the same current and loss to heat. Rewriting the power balance as
$$I\mathcal{E} = I^2R - I\mathcal{E}_F,$$
show that $-I\mathcal{E}_F$ exactly matches the P_{rot} you calculated earlier.