Assignment 10

Due date: Friday, November 19

Electric field in a moving frame

In a frame with only static charge, there is a region of uniform electric field

$$\mathbf{E} = E\,\hat{\mathbf{x}}.$$

What electric field $\mathbf{E}'$ is seen by an observer moving with velocity

$$\mathbf{v} = \frac{c}{2}(\hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}})$$

with respect to this frame (and in the same region)?

If you are experiencing *dèjá vu* right now it is because essentially this same problem was given on prelim 2. What the authors of the exam expected to be 20 free points for doing routine math turned out to be nothing of the sort! Here is your second chance and to ensure success this time, here is a checklist:

- Write explicit expressions for $\mathbf{E}_{||}$ and $\mathbf{E}_{\perp}$ in terms of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$.
- Check that your $\mathbf{E}_{||}$ is parallel to $\mathbf{v}$, and that $\mathbf{v} \cdot \mathbf{E}_{\perp} = 0$.
- Check that $\mathbf{E}_{||} + \mathbf{E}_{\perp} = \mathbf{E}$.
- When computing the Lorentz contraction factor $\gamma = 1/\sqrt{1 - v^2/c^2}$ be sure to use $v^2 = \mathbf{v} \cdot \mathbf{v}$.

Electric and magnetic fields in compound Lorentz transformations

In this problem we consider the transformation of electric and magnetic fields for frames related by boosts with relative velocities $\pm\mathbf{v}$. As usual, we use subscripts $||$ and $\perp$ to denote, respectively, the parts of a vector parallel and perpendicular to $\mathbf{v}$. In the un-primed frame there is only an electric field and it is perpendicular to $\mathbf{v}$. That is, $\mathbf{E} = \mathbf{E}_{\perp}$ and $\mathbf{B} = 0$.

(a) The primed frame has velocity $+\mathbf{v}$ relative to the un-primed frame. Using the general field transformation rules derived in lecture, determine the fields $\mathbf{E}'$ and $\mathbf{B}'$ in the primed frame in terms of the fields in the un-primed frame (just the electric field $\mathbf{E}_{\perp}$) and the velocity $\mathbf{v}$.

(b) Explain why a nonzero $\mathbf{B}'$ in the primed frame is not in conflict with the lecture derivation of $\mathbf{E}'_\perp = \gamma \mathbf{E}_\perp$.

Hint: Recall the velocity of the charge in the primed frame and how that relates to the force law.)

(c) The double-primed frame has velocity $-\mathbf{v}$ relative to the primed frame. Using the general field transformation rules derived in lecture, determine the fields $\mathbf{E}''$ and $\mathbf{B}''$ in the double-primed frame. Since in (a) you expressed primed-frame fields in terms of the un-primed fields, you should express $\mathbf{E}''$ and $\mathbf{B}''$ just in terms of $\mathbf{E}_\perp$ and the velocity $\mathbf{v}$.

(d) Is your answer to (c) the one you were expecting?

Magnetic dipoles

Since (as far as anyone knows) there are no magnetic monopoles, the simplest microscopic entity that can be a source of magnetic field and also be acted upon by magnetic fields is the magnetic dipole. Microscopic magnetic dipoles that occur in materials can be modeled as closed loops of current which maintain a fixed value of current I in circulation about the loop. The mechanism that keeps the current fixed is beyond the scope of this course, as it involves quantum mechanics.

The current loops C you will be considering in this problem are always planar but otherwise arbitrary (a circle, rectangle, etc.).

1. Prove the following geometrical identity for planar loops:

$$\oint_C \mathbf{r} \cdot \mathbf{V} \, d\mathbf{r} = \mathcal{A} \times \mathbf{V}. \tag{1}$$

Here $\mathbf{V}$ is a constant vector (not necessarily in the same plane as C) and $\mathcal{A}$ is defined as the vector perpendicular to the plane of C whose magnitude equals the area enclosed by C. The right-hand-rule applies to the orientation of $\mathcal{A}$, so if C circulates counter-clockwise in the x-y plane, $\mathcal{A}$ is in the positive z direction.

Hint: Work out the integral in a coordinate system where C is in the x-y plane and then interpret your result in terms of the vector expression (1) that applies in any coordinate system. There will be two integrals, since (in your coordinate system) the line element $d\mathbf{r}$ has both an x component and a y component. **If you get stuck, at least work out the special case where C is a rectangle.**

2. In this part you are interested in the forces on the current loop in the presence of a uniform external magnetic field $\mathbf{B}$. Starting from the force

$$d\mathbf{F} = I d\mathbf{r} \times \mathbf{B}$$

on an element $d\mathbf{r}$ of the loop, first show that the net magnetic force vanishes. Next find the torque on the loop by calculating the line integral

$$\mathbf{T} = \oint_C \mathbf{r} \times d\mathbf{F}.$$

After using the double cross-product identity you will get two integrals, one of which is identically zero and the other is an instance of the identity you proved above. Your answer should have the form

$$\mathbf{T} = \mathbf{m} \times \mathbf{B},$$

where the vector $\mathbf{m}$ is proportional to the area vector $\mathcal{A}$.

3. In this last part there is no *external* magnetic field because you are interested in the loop as a *source* of magnetic field. Start with the formula for the vector potential $\mathbf{A}$ we derived in lecture:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint_C \frac{I\,d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}.$$

The magnetic field only has the dipole form when the field point $\mathbf{r}$ is far from the loop. You should therefore use just the first two terms in the expansion

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} + \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{r^2} \cdots \ .$$

As in the previous part, one integral will be identically zero and in the other you have another opportunity to use the geometrical identity (1). Express your answer for $\mathbf{A}$ in terms of the same vector $\mathbf{m}$ that appears in the torque formula. You are being spared the mind-numbing mechanical exercise of calculating $\mathbf{B}$ from $\mathbf{A}$ by taking the curl!