Assignment 1

Due date: Wednesday, February 27

The number of partitions from their shape

In lecture we found the *shape* of typical partitions of N, in the limit of large N, by maximizing entropy subject to a constraint. In rescaled coordinates, the curve for this shape (in the positive quadrant) satisfies

$$e^{-x} + e^{-y} = 1.$$

1. Find the rescaling factor so that x counts the "parts" and $y(x)$ gives the "size of part x". The area under the curve, after rescaling, should be N.
2. Evaluate the entropy, with your rescaling, and estimate the number of partitions of N. Your answer (to leading order) should agree with the result of the (very different!) saddle point calculation from lecture.

Square grid of resistors

In an infinite square grid of 1 Ohm resistors, let $R(m,n)$ be the equivalent resistance between node $(0,0)$ and node (m,n), where m and n are integers.

1. Using Fourier methods show that

$$R(m,n) = \int_{-\pi}^{\pi} \frac{dp}{2\pi} \int_{-\pi}^{\pi} \frac{dq}{2\pi} \; \frac{1-\cos(mp+nq)}{2-\cos p - \cos q}.$$

2. The grid should behave like a continuum when $\sqrt{m^2+n^2} = l \to \infty$, and we expect R to only depend on l in this limit. Find the leading $l \to \infty$ behavior of R by the following steps:

   (a) Using the following definitions,

$$m = l\cos\theta$$
$$n = l\sin\theta$$
$$p = r\cos\phi$$
$$q = r\sin\phi,$$

   replace $R(m,n)$ by its angular average, $R(l) = \langle R(m,n)\rangle_\theta$.

(b) Express $R(l) = R_< + R_>$ by partitioning the (r, ϕ) integral into two regions,

$$R_< : \; r < \epsilon$$
$$R_> : \; r > \epsilon,$$

where ϵ is defined such that in the $l \to \infty$ limit, $\epsilon \to 0$ while also $\epsilon l \to \infty$ (e.g. $\epsilon \sim 1/\sqrt{l}$). You will find that it is easy to obtain both $R_<$ and $R_>$ to leading order[1], and that both depend on ϵ. However, their sum is independent of ϵ to leading order, as it should be.

[1] In particular, the Bessel function produced by the angular average can be neglected.