Due date: Thursday, September 12

This first homework assignment is meant to get you up to speed on the hard matrix model (HMM)[1]. Much of this can be found online but you are encouraged to work out everything from scratch.

1. An $n \times n$ *Hadamard matrix* H has only ± 1 elements and orthogonal rows. Prove that the columns are also orthogonal. *Hint:* Relate the transpose to the inverse and the use the equality of right/left inverses.

2. Write down examples of 1×1 and 2×2 Hadamard matrices. Show that all higher $n \times n$ Hadamard matrices must have n divisible by 4. *Hint:* By multiplying rows by -1, as necessary, you can arrange to have all rows start with $+\,+\,+$, $+\,+\,-$, $+\,-\,+$, or $+\,-\,-$. Say there are $n_1, \ldots, n_4$ rows with those starts. Now use orthogonality of the first three columns to find relations among these integers.

3. Go to sequence A206711 of OEIS to get counts of Hadamard matrices up to $n = 32$. By analyzing these numbers, speculate about the *entropy* of Hadamard matrices. You might want to first factor out the symmetry group (flipping the sign of any row or column, or any row/column permutation, gives another Hadamard). Entropy in thermodynamics is extensive, but in this case it's not clear what takes the place of "volume" — is it n or n^2, or something else? There's no "correct" answer to this problem, its aim is just to get you thinking.

4. The Hadamard matrices of order n are special points in the manifold of orthogonal matrices U (with $\sqrt{n}$ convention for the norms of the rows). Here is a Hamiltonian designed to single out Hadamard matrices as the ground states in this manifold,

$$\mathcal{H}(U) = -\sum_{i=1}^{n}\sum_{j=1}^{n} |U_{ij}|^{\alpha},$$

where $\alpha > 0$ is a parameter. Note that $\mathcal{H}(U)$ is a constant (trivial) for $\alpha = 2$. Unless stated otherwise, we will usually take $\alpha = 1$. From a

[1]The 2019 students of 7653 thought this was a good name: a matrix model that's surprisingly hard.

computational perspective this is the least expensive way of having a cusp at zero (incentivizing matrix elements to select a sign). This is the hard matrix model.

Use the generalized mean inequality to prove that the ground states of $\mathcal{H}$, for $\alpha < 2$, are Hadamard matrices for those n where Hadamard matrices exist.

5. Find the HMM ($\alpha = 1$) ground states for $n = 3$.

6. We will study the Gibbs probability density

$$dU \exp\left(-\beta\, \mathcal{H}(U)\right),$$

where dU is the uniform measure on the orthogonal matrices. You can think of dU as follows. Fix $n - 2$ rows of U. The remaining two rows are orthogonal to all of these and each other. This means they have one continuous degree of freedom: the rotation in the plane they span. The uniform measure dU corresponds to all rotations (of the vector-pair) having the same probability. Given[2] this, and the fact that rotations applied to all pairs of rows generate the entire set of orthogonal matrices[3], we now have a way to use Metropolis-Hastings to sample the Gibbs distribution for the HMM.

Add enough comments to the C function `rowrot(i,j,a)` to convince the grader that you understand how it is applying the Metropolis-Hastings transition rule to rows `i` and `j` with rotation angle `a`.

[2]Transformations involving just a pair of rows are called *Givens rotations*, after Wallace Givens.

[3]We would have to add improper rotations as well, if we wanted to sample the complete set of orthogonal matrices. But this doesn't add anything interesting.