Particle position distribution in a 3D potential

The probability density $\rho(\boldsymbol{r})$ is proportional to the number of microstates at $\boldsymbol{r}$. This is given by the number of momentum microstates at that position, which is the volume of a spherical shell of constant energy divided by the volume of one momentum microstate $\delta p_x \delta p_y \delta p_z$. In lecture it was shown that the volume of this shell is $4\pi\sqrt{2mK}m\delta E$, so the number of microstates is $\frac{4\pi m\sqrt{2mK}\delta E}{\delta p_x \delta p_y \delta p_z}$, so $\rho(\boldsymbol{r}) = C\sqrt{K(\boldsymbol{r})}$, where C is a normalization constant. The kinetic energy is $K = E - U(\boldsymbol{r})$, so $\rho(\boldsymbol{r}) = C\sqrt{E - U(\boldsymbol{r})}$. From the normalization condition, we have

$$\int C\sqrt{E - U(\boldsymbol{r})}d^3\boldsymbol{r} = 1 \rightarrow C = \frac{1}{\int \sqrt{E - U(\boldsymbol{r})}d^3\boldsymbol{r}}.$$

This finally gives

$$\rho(\boldsymbol{r}) = \frac{\sqrt{E - U(\boldsymbol{r})}}{\int \sqrt{E - U(\boldsymbol{r})}d^3\boldsymbol{r}}.$$

Tracer particle analysis of soft billiards

The time to cross the circular region at a vertical distance y from the center is simply the length of the chord at that height divided by the speed, so $t_1(y) = \frac{Length\ of\ chord\ at\ y}{v_1}$. From the Pythagorean theorem, half the length of the chord is $\sqrt{r^2 - y^2}$, so the length of the chord is $2\sqrt{r^2 - y^2}$. This gives $t_1(y) = \frac{Length\ of\ chord\ at\ y}{v_1} = \frac{2\sqrt{r^2-y^2}}{v_1}$. The total time is then $T_1 = \int_{-r}^{r} t_1(y)\rho dy = \frac{\rho}{v_1}\int_{-r}^{r} 2\sqrt{r^2 - y^2}dy$. The integral is the area of a circle, so

$$T_1 = \frac{\rho}{v_1}\pi r^2.$$

The time is, as before, the length of the chord divided by the speed. The length of half the chord is $r\cos(\theta_2)$, for $-\frac{\pi}{2} < \theta_2 < \frac{\pi}{2}$, since outside this angle range, the particles are reflected and do not enter the circular region, and therefore the chord length is 0. The total length of the chord is $2r\cos(\theta_2) = 2r\sqrt{1 - \sin^2(\theta_2)}$. Using $v_1\sin(\theta_1) = v_2\sin(\theta_2)$, Schnell's law, we have $\sin(\theta_2) = \frac{v_1}{v_2}\sin(\theta_1)$, so the length of the chord is $2r\sqrt{1 - \left(\frac{v_1}{v_2}\right)^2\sin^2(\theta_1)}$. Note that the condition $-\frac{\pi}{2} \leq \theta_2 \leq \frac{\pi}{2}$ implies $-1 < \frac{v_1}{v_2}\sin(\theta_1) < 1$, or $-\frac{v_2}{v_1} < \sin(\theta_1) < \frac{v_2}{v_1}$. From the definition of the sine function, we see $\sin(\theta_1) = \frac{y}{r}$, so the length of the chord is $2r\sqrt{1 - \left(\frac{v_1}{v_2}\right)^2\left(\frac{y}{r}\right)^2}$ for $-\frac{v_2}{v_1} < \frac{y}{r} < \frac{v_2}{v_1}$, or $|y| < \frac{v_2}{v_1}r$, and 0 otherwise. The time is then

$$t_2(y) = \frac{2r}{v_2}\sqrt{1 - \left(\frac{v_1}{v_2}\right)^2\left(\frac{y}{r}\right)^2}, |y| < \frac{v_2}{v_1}r$$

And the total time is

$$T_2 = \frac{2r}{v_2}\rho\int_{-\frac{v_2 r}{v_1}}^{\frac{v_2 r}{v_1}}\sqrt{1-\left(\frac{v_1}{v_2}\right)^2\left(\frac{y}{r}\right)^2}\,dy = \frac{\rho}{v_2}\int_{-\frac{v_2 r}{v_1}}^{\frac{v_2 r}{v_1}} 2\sqrt{r^2-\left(\frac{v_1}{v_2}\right)^2 y^2}dy.$$

Changing variables to $u = \frac{v_1}{v_2}y, du = \frac{v_1}{v_2}dy$, this becomes

$$T_2 = \frac{\rho}{v_1}\int_{-r}^{r} 2\sqrt{r^2-u^2}du = \frac{\rho}{v_1}\pi r^2 = T_1.$$

Monoatomic and diatomic gases in thermal contact

1. For the monoatomic gas we can simply repeat the calculation in lecture and obtain
$$\Omega(V_1,E_1) \propto V_1^{N_1}E_1^{\frac{3N_1}{2}-1}.$$
2. The entropy is $S_1 = k_B\ln(\Omega(V_1,E_1)) = k_B\ln\left(V_1^{N_1}E_1^{\frac{3N_1}{2}-1}\right) = k_B\left(N_1\ln(V_1)+\frac{3N_1}{2}\ln(E_1)\right)+C_1$, where we have used $\frac{3N_1}{2}-1 \approx \frac{3N_1}{2}$.
3. The temperature is given by $T_1 = \frac{1}{\frac{dS_1}{dE_1}} = \frac{1}{\frac{3N_1 k_B}{2\ E_1}} = \frac{2E_1}{3N_1 k_B}$.
4. The position microstates are the same, but now the energy is $E_2 = \frac{1}{2m_2}\left(\sum_{i=1}^{N_2} p_{xi}^2 + p_{yi}^2 + p_{zi}^2\right) + \frac{1}{2I}\left(\sum_{i=1}^{N_2} L_{xi}^2 + L_{yi}^2\right)$. In each of the p dimensions the "radius" of the hypersphere shell is $R = \sqrt{2m_2E_2}$ and in each L dimension it is $\sqrt{2IE_2}$. The volume of the hypersphere (hyper-ellipse actually) is then proportional to $\sqrt{E_2}^{5N_2}$, so the volume of the hypershell is
$$\Omega(V_2,E_2) \propto V_2^{N_2}E_2^{\frac{5N_2}{2}-1} \approx V_2^{N_2}E_2^{\frac{5N_2}{2}}.$$
5. The entropy is $S_1 = k_B\ln(\Omega(V_2,E_2)) = k_B\ln\left(V_2^{N_2}E_2^{\frac{3N_2}{2}-1}\right) = k_B\left(N_2\ln(V_2)+\frac{5N_2}{2}\ln(E_2)\right)+C_2$.
6. The temperature is $T_2 = \frac{1}{\frac{dS_2}{dE_2}} = \frac{1}{\frac{5N_2 k_B}{2\ E_2}} = \frac{2E_2}{5N_2 k_B}$.
7. When placed in thermal contact, then we have $T_{1,new} = T_{2,new}$ which means
$$\frac{2(E_1+\Delta E)}{3N_1k_B} = \frac{2(E_2-\Delta E)}{5N_2k_B}$$
$$T_1 + \frac{2\Delta E}{3N_1k_B} = T_2 - \frac{2\Delta E}{5N_2k_B}$$
$$\Delta E = \frac{k_B(T_2-T_1)}{2\left(\frac{1}{3N_1}+\frac{1}{5N_2}\right)}.$$
8. ΔE is proportional to $(T_2 - T_1)$, which implies it is positive for $T_2 > T_1$, consistent with heat flowing from hot to cold.