Euler's partial derivative product identity

Here's an identity involving partial derivatives that comes up in the first-principles derivation of the ideal gas law:

$$\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial z}{\partial x}\right|_y \left.\frac{\partial y}{\partial z}\right|_x = -1\ . \tag{1}$$

Notice the cyclic symmetry of the symbols: $x \to y \to z \to x$. These notes are meant to help you understand what the identity *means*, and how it is derived using the math you learned in multi-variable calculus and linear algebra. Together we can lament the fact that this identity is left out of the standard curriculum, and that its Wikipedia page is terrible.

A symmetrical way to express the fact that any two of a set of three variables determines the third, is to say there exists a function of three variables f such that

$$f(x, y, z) = 0 \tag{2}$$

defines that relationship. For this to be a useful characterization at more than a single point (x, y, z), the set defined by equation (2) should be a smooth surface, and that implies the gradient of f, the vector

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\ ,$$

is never zero when evaluated at points of the surface. This is because the space orthogonal to ∇f defines the tangent plane to the surface.

The statement that x is determined by y and z is made more explicit by writing (2) in the form

$$f(x(y, z), y, z) = 0\ , \tag{3}$$

which defines x as a function of the two independent variables y and z. The analogous statements for y and z are

$$f(x, y(z, x), z) = 0 \tag{4}$$
$$f(x, y, z(x, y)) = 0\ . \tag{5}$$

We get Euler's identity by taking partial derivatives. Taking the partial derivative of (3) with respect to y and using the chain rule, we obtain

$$\frac{\partial f}{\partial x}\left(\left.\frac{\partial x}{\partial y}\right|_z\right) + \frac{\partial f}{\partial y} = 0\ . \tag{6}$$

The $|_z$ reminds us that z is kept constant as we take the partial derivative of $x(y, z)$ with respect to y. We do not need this kind of reminder in the case of derivatives of f if we understand that $\partial f/\partial x$ means "take the derivative of the first argument of f ".

Continuing cyclically, partial derivative of (4) with respect to z, and partial derivative of (5) with respect to x, we get two cyclically related equations:

$$\frac{\partial f}{\partial y}\left(\left.\frac{\partial y}{\partial z}\right|_x\right) + \frac{\partial f}{\partial z} = 0 \tag{7}$$

$$\frac{\partial f}{\partial z}\left(\left.\frac{\partial z}{\partial x}\right|_y\right) + \frac{\partial f}{\partial x} = 0 \, . \tag{8}$$

Equations (6), (7) and (8) can be written compactly as a single matrix equation,

$$\nabla f \cdot \mathbf{E} = 0 \tag{9}$$

where ∇f is the row vector of partial derivatives, and

$$\mathbf{E} = \begin{bmatrix} \left.\frac{\partial x}{\partial y}\right|_z & 0 & 1 \\ 1 & \left.\frac{\partial y}{\partial z}\right|_x & 0 \\ 0 & 1 & \left.\frac{\partial z}{\partial x}\right|_y \end{bmatrix} \tag{10}$$

is the "Euler matrix." As argued earlier, ∇f is not the zero vector, so that equation (9) implies the the Euler matrix does not have full rank (it has a nontrivial null space). But that implies the determinant of $\mathbf{E}$ must be zero. Working out the determinant and setting it to zero we obtain the identity (1).